Subject: Heat Transfer-I

Dr. Mustafa B. Al-hadithi

First Lecture

Introduction to Heat Transfer

1- What is Heat Transfer?

"Energy in transit due to temperature difference."

Thermodynamics tells us:

- How much heat is transferred (δQ)
- How much work is done (δW)
- Final state of the system

Heat transfer tells us:

- How (with what modes) δQ is transferred
- At what rate δQ is transferred
- Temperature distribution inside the body

Heat transfer ⟷ complementary ⟷ Thermodynamics

2- Heat transfer Modes.

✓ Conduction
- needs matter
- molecular phenomenon (diffusion process)
- without bulk motion of matter

✓ Convection
- heat carried away by bulk motion of fluid
- needs fluid matter

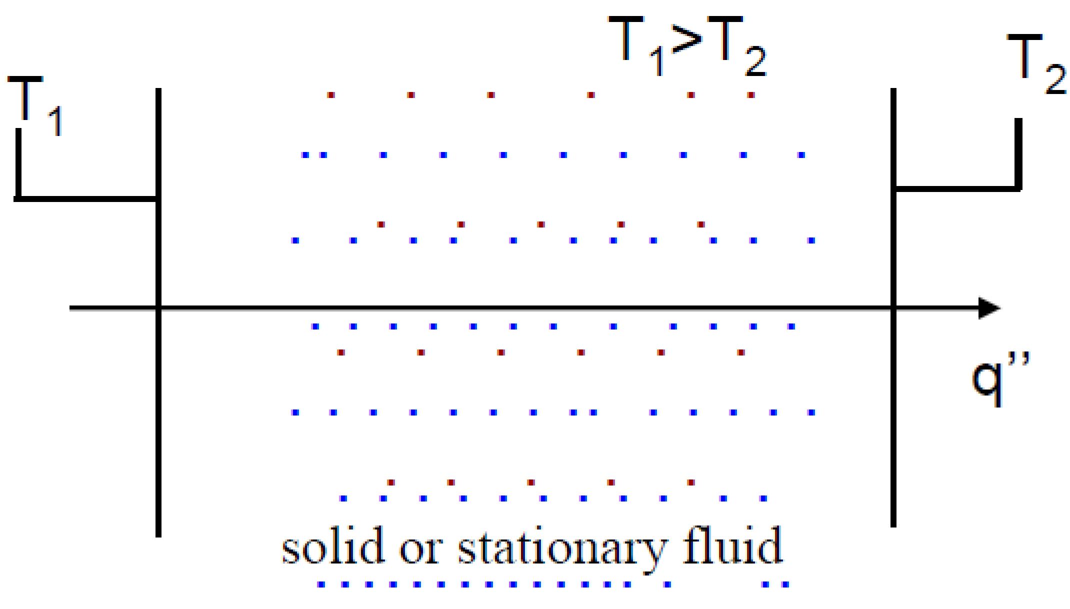
✓ Radiation
- does not needs matter
- transmission of energy by electromagnetic waves

3- Applications of Heat Transfer.

- ✓ Energy production and conversion
  - steam power plant, solar energy conversion etc.
- ✓ Refrigeration and air-conditioning
- ✓ Domestic applications
  - ovens, stoves, toaster
- ✓ Cooling of electronic equipment
- ✓ Manufacturing / materials processing
  - welding,casting, soldering, laser machining
- ✓ Automobiles / aircraft design
- ✓ Nature (weather, climate etc)

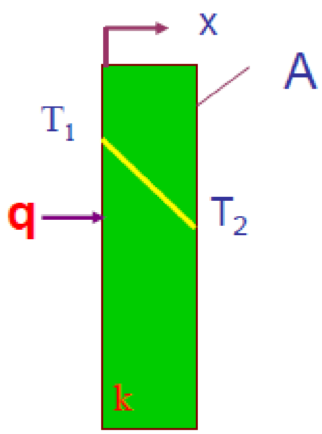
4- Heat Transfer by Conduction Mode.

In this mode type of heat transfer required (**Medium and Temperature Gradient**)



RATE:

q(W) or (J/s)   (heat flow per unit time)



Rate equations (1D conduction):

- Differential Form

  $q = - k\ A\ dT/dx$, W

  k = Thermal Conductivity, W/mK

  A = Cross-sectional Area, m^2

  T = Temperature, K or °C

  x = Heat flow path, m

- Difference Form

  $q = k\ A\ (T_1 - T_2) / (x_1 - x_2)$

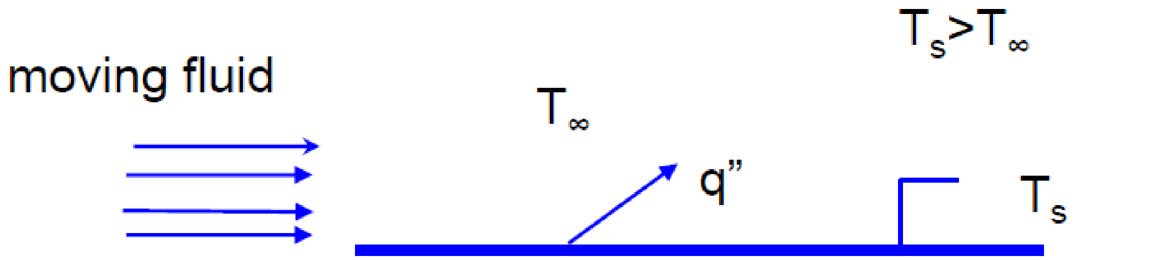
Heat flux: $q'' = q / A = - kdT/dx$ (W/m^2)

(negative sign denotes heat transfer in the direction of decreasing temperature)

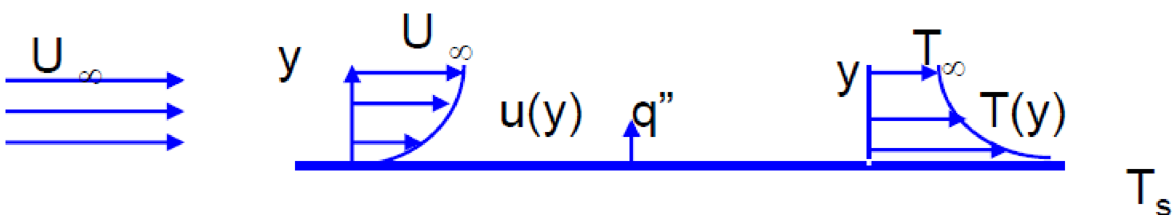
- Example:

The wall of an industrial furnace is constructed from 0.15 m thick fireclay brick having a thermal conductivity of 1.7 W/mK. Measurements made during steady state operation reveal temperatures of 1400 and 1150 K at the inner and outer surfaces, respectively. What is the rate of heat loss through a wall which is 0.5 m by 3 m on a side ?

5- Heat Transfer by Convection Mode.



- Energy transferred by diffusion + bulk motion of fluid

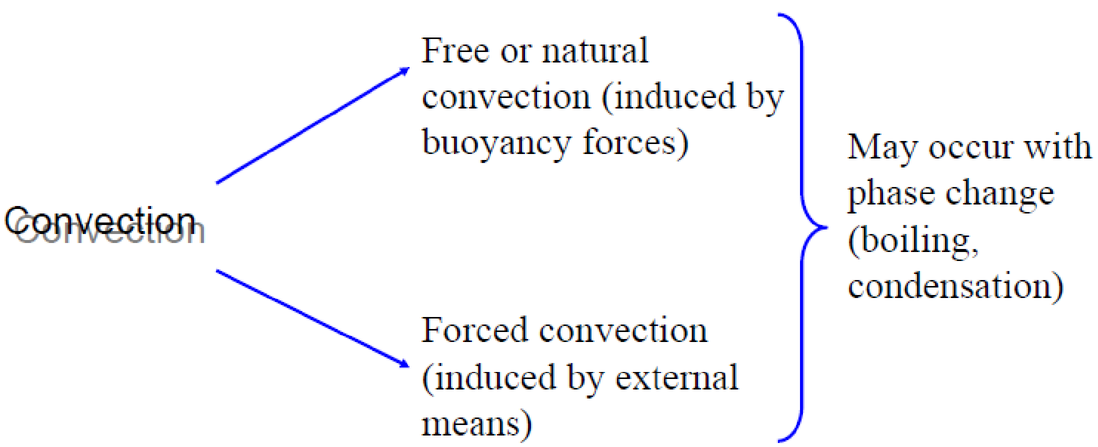


Heat transfer rate $q = hA(T_s - T_\infty)$ W

Heat flux $q'' = h(T_s - T_\infty)$ W / m^2

h=heat transfer co-efficient (W /m^{2}K)

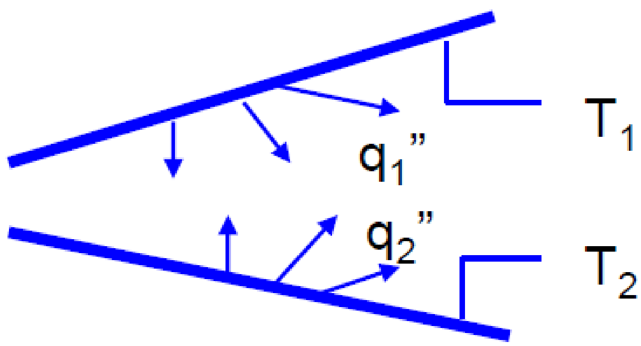
The properties depends on geometry, nature of flow, thermodynamics properties etc.



Typical values of h (W/m^2K)

Free convection	gases: 2 - 25
	liquid: 50 - 100
Forced convection	gases: 25 - 250
	liquid: 50 - 20,000
Boiling/Condensation	2500 -100,000

6- **Heat Transfer by Radiation Mode.**



RATE:

q(W) or (J/s) Heat flow per unit time.

Flux : q'' (W/m^2)

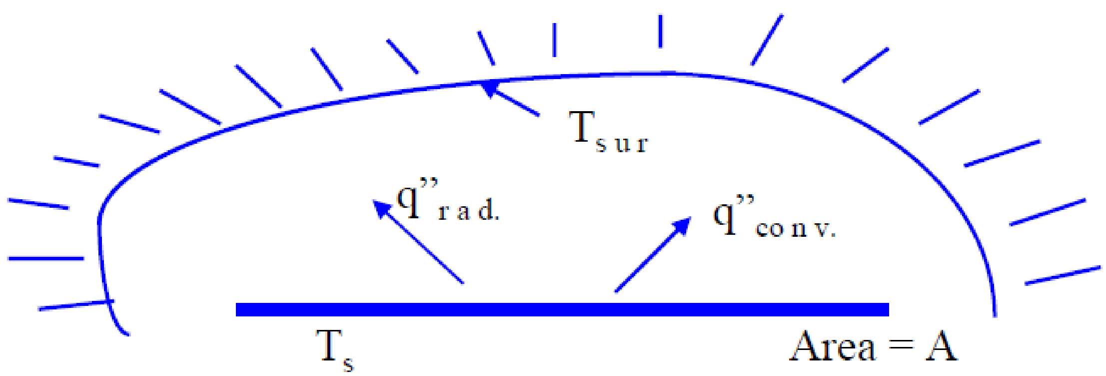
Heat Transfer by electro-magnetic waves or photons(no medium required.)

Emissive power of a surface (energy released per unit area):

$$E=\varepsilon\sigma T_s^4 \ (W/\ m^2)$$

ε= emissivity (property)………

σ=Stefan-Boltzmann constant



Radiation exchange between a large surface and surrounding

$$Q''_{rad} = \varepsilon\sigma(T_s^4 - T_{sur}^4)\ W/\ m^2$$

- **Example:**

An uninsulated steam pipe passes through a room in which the air and walls are at 25 °C. The outside diameter of pipe is 70 mm, and its surface temperature and emissivity are 200 °C and 0.8, respectively. If the coefficient associated with free convection heat transfer from the surface to the air is 5 W/m^2K, what is the rate of heat loss from the surface per unit length of pipe ?

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Second Lecture

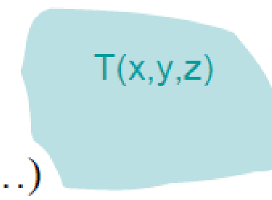
One Dimentional Steady State Heat Conduction

1- Objectives of Conduction Analysis.

To determine the temperature field, T(x,y,z,t), in a body (i.e. how temperature varies with position within the body)

❑T(x,y,z,t) depends on:

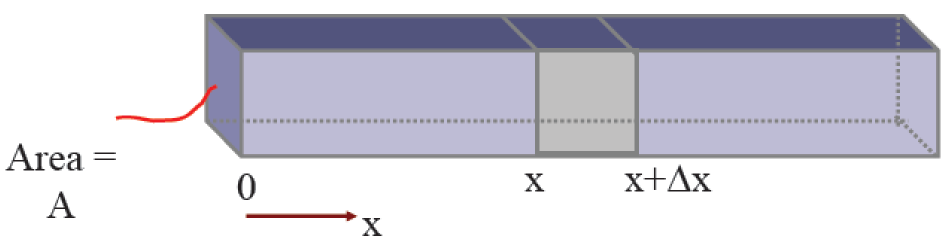
- boundary conditions
- initial condition
- material properties (k, c^p, ρ …)
- geometry of the body (shape, size)



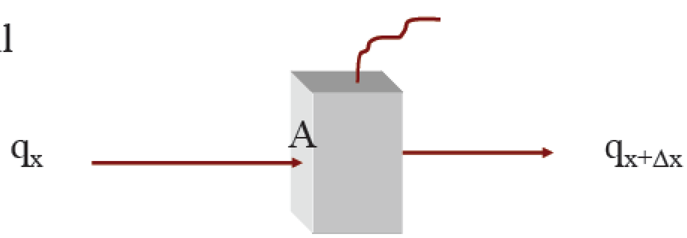
❑Why we need T(x,y,z,t) ?

- to compute heat flux at any location (using Fourier's eqn.)
- compute thermal stresses, expansion, deflection due to temp. etc.
- design insulation thickness
- chip temperature calculation
- heat treatment of metals

2- One Dimension Heat Conduction.



Solid bar, insulated on all long sides (1D heat conduction)



$\dot{q}$ = Internal heat generation per unit vol. (W/m^3)

First Law (energy balance) $(\dot{E}_{in} - \dot{E}_{out}) + \dot{E}_{gen} = \dot{E}_{st}$

$$q_x - q_{x+\Delta x} + A(\Delta x)\dot{q} = \frac{\partial E}{\partial t}$$

$$E = (\rho A \Delta x)u$$

$$\frac{\partial E}{\partial t} = \rho A \Delta x \frac{\partial u}{\partial t} = \rho A \Delta x c \frac{\partial T}{\partial t}$$

$$q_x = -kA\frac{\partial T}{\partial x}$$

$$q_{x+\Delta x} = q_x + \frac{\partial q_x}{\partial x}\Delta x$$

$$-kA\frac{\partial T}{\partial x} + kA\frac{\partial T}{\partial x} + A\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)\Delta x + A\Delta x\dot{q} = \rho A c \Delta x \frac{\partial T}{\partial t}$$

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \dot{q} = \rho c \frac{\partial T}{\partial t}$$

Longitudinal conduction

Internal heat generation

Thermal inertia

If k is a constant

$$\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{k} = \frac{\rho c}{k}\frac{\partial T}{\partial t} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

- ❑ For T to rise, LHS must be positive (heat input is positive)
- ❑ For a fixed heat input, T rises faster for higher α
- ❑ In this special case, heat flow is 1D. If sides were not insulated, heat flow could be 2D, 3D.

3- Boundary and Initial Conditions.

- The objective of deriving the heat diffusion equation is to determine the temperature distribution within the conducting body.

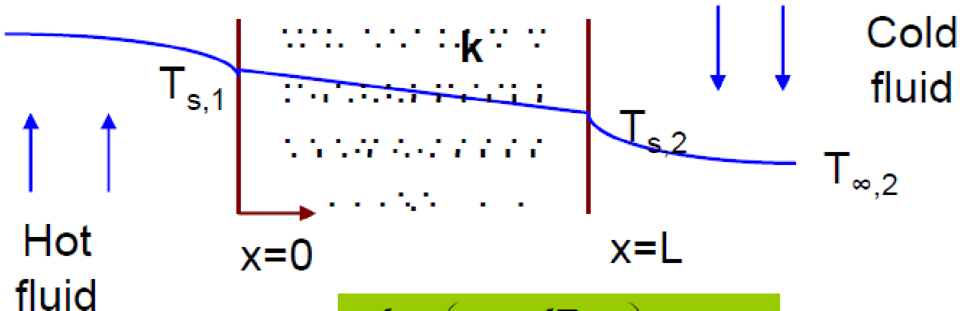
- We have set up a differential equation, with T as the dependent variable. The solution will give us T(x,y,z). Solution depends on boundary conditions (BC) and initial conditions (IC).

How many BC's and IC's ?

- Heat equation is second order in spatial coordinate. Hence, 2 BC's needed for each coordinate.
  - * 1D problem: 2 BC in x-direction
  - * 2D problem: 2 BC in x-direction, 2 in y-direction
  - * 3D problem: 2 in x-dir., 2 in y-dir., and 2 in z-dir.
- Heat equation is first order in time. Hence one IC needed

4- Plan Wall Heat Conduction.

The Plane Wall :



$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0$$

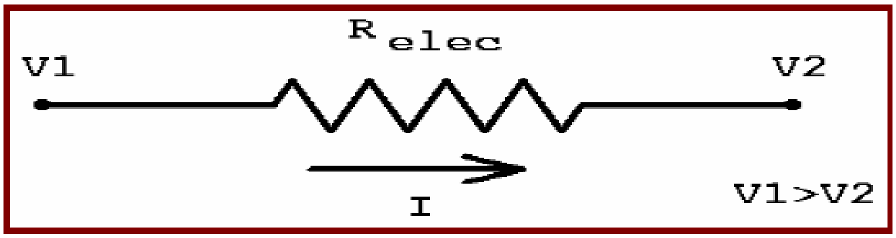
Const. K; solution is:

$$q_x = -kA\frac{dT}{dx} = \frac{kA}{L}\left(T_{s,1} - T_{s,2}\right) = \frac{T_{s,1} - T_{s,2}}{L/kA}$$

5- <u>Thermal Resistance (Electrical Analogy).</u>

OHM's LAW :Flow of Electricity

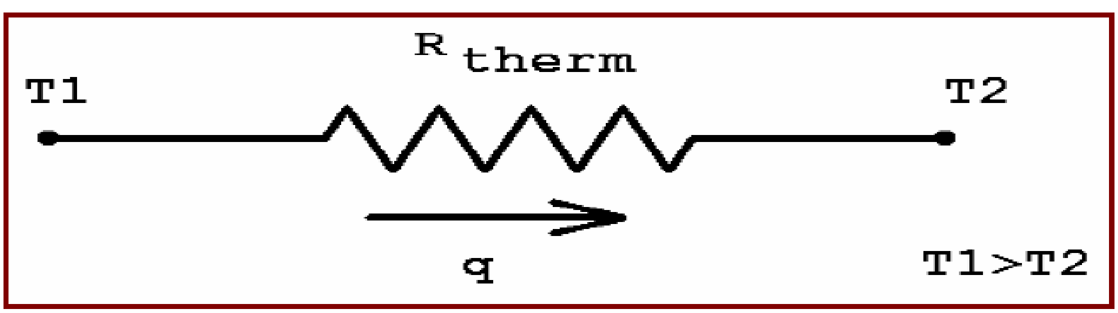
$$V = IR_{elect}$$

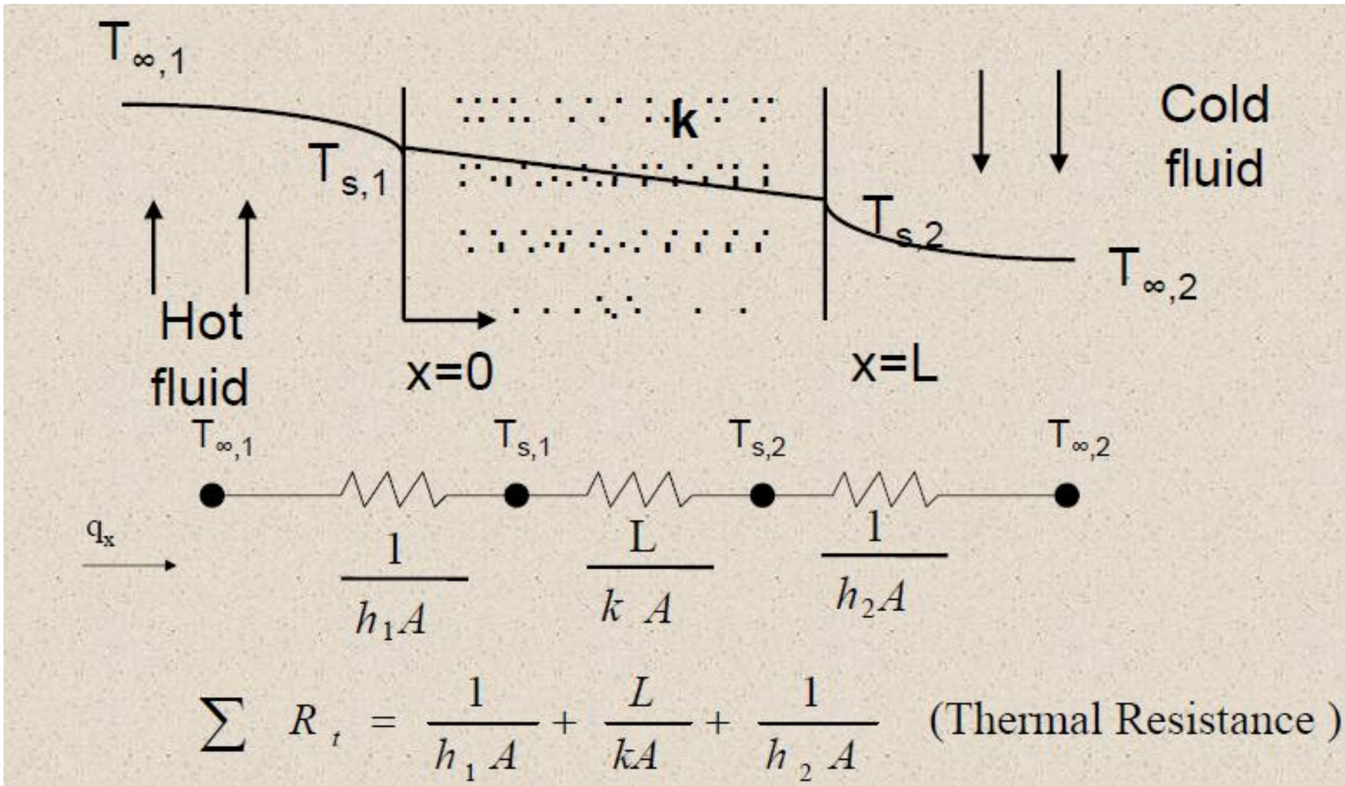


Voltage Drop = Current flow×Resistance

$$\Delta T = qR_{therm}$$

Temp Drop=Heat Flow×Resistance





THERMAL RESISTANCES

- Conduction

  $R_{cond} = \Delta x/kA$

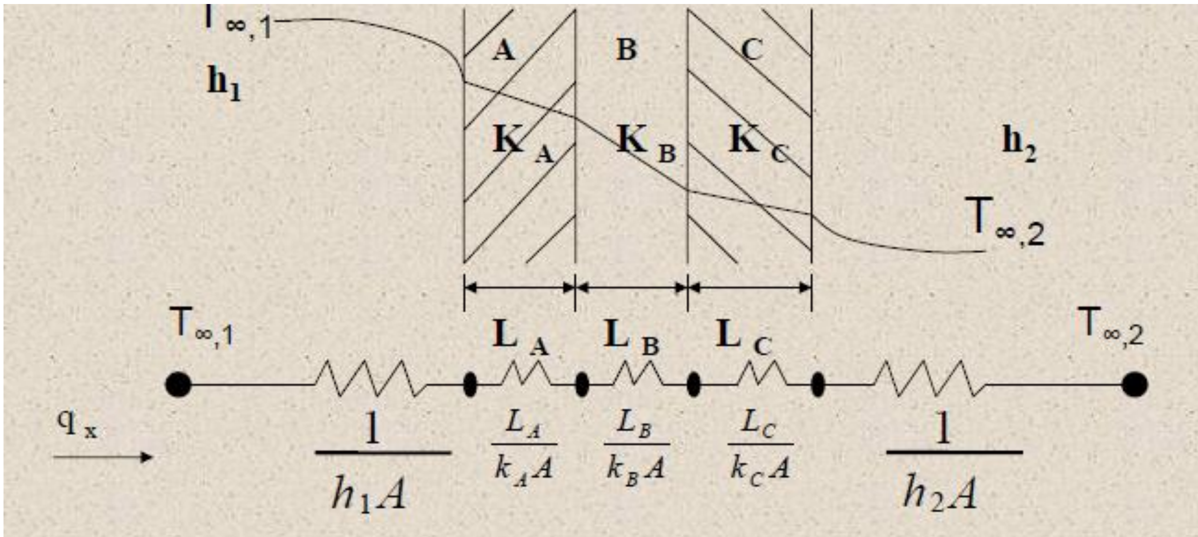
- Convection

  $R_{conv} = (hA)^{-1}$

- Fins

  $R_{fin} = (h\eta A)^{-1}$

6- Composit Wall.



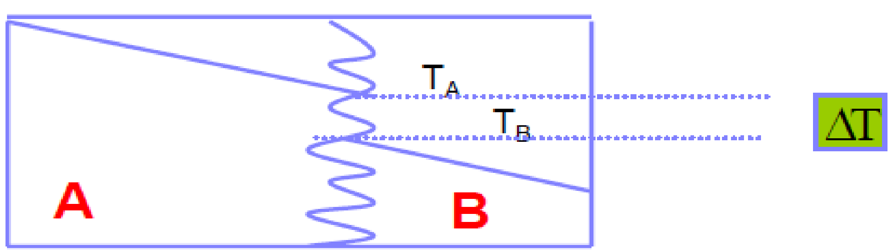
$$q_x = \frac{T_{\infty,1} - T_{\infty,2}}{\sum R_t} = \frac{T_{\infty,1} - T_{\infty,2}}{\frac{1}{h_1 A} + \frac{L_A}{k_A} + \frac{L_B}{k_B} + \frac{L_C}{k_C} + \frac{1}{h_2 A}} = UA \ \Delta T$$

$$where, U = \frac{1}{R_{tot} A} = \text{Overall heat transfer coefficient}$$

7- Overall Heat Transfer Coefficient.

$$U = \frac{1}{R_{total} A} = \frac{1}{\frac{1}{h_1} + \Sigma \frac{L}{k} + \frac{1}{h_2}}$$

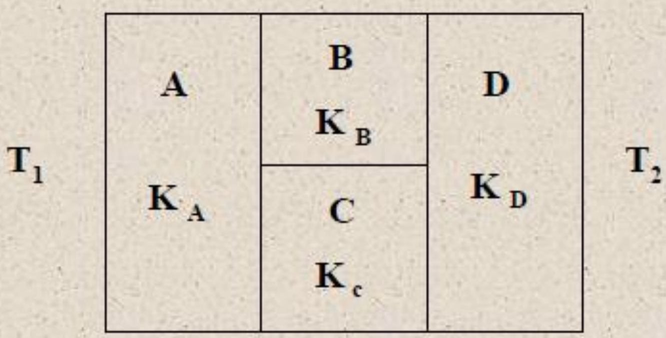
Contact Resistance :



$$R_{t,c} = \frac{\Delta T}{q_x}$$

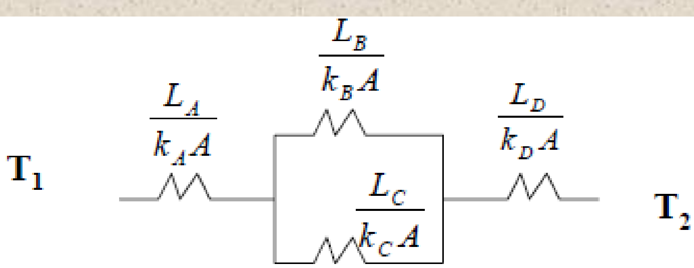
$$U = \frac{1}{\frac{1}{h_1} + \frac{L_A}{k_A} + \frac{L_B}{k_B} + \frac{L_C}{k_C} + \frac{1}{h_2}}$$

Series-Parallel :



$A_B + A_C = A_A = A_D$

$L_B = L_C$



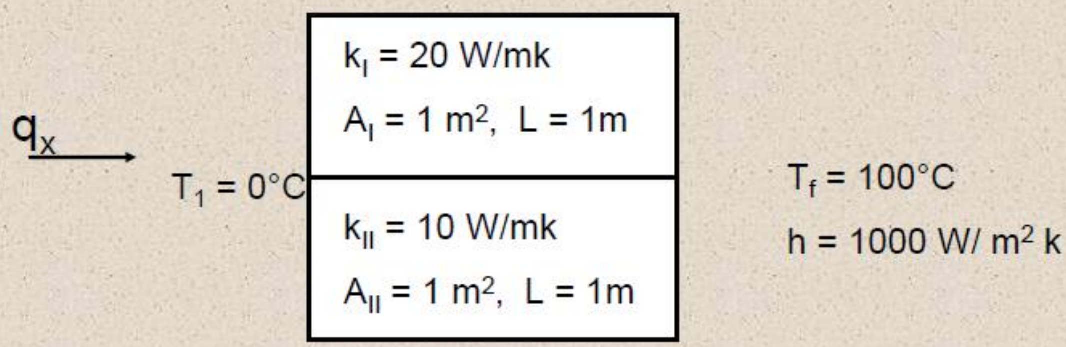
Assumptions :

(1) Face between B and C is insulated.

(2) Uniform temperature at any face normal to X.

Example:

Consider a composite plane wall as shown:



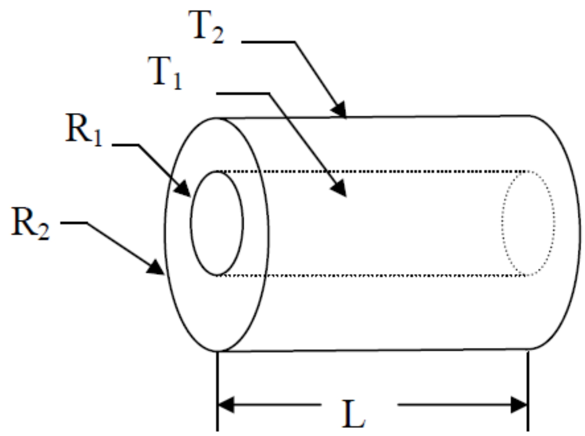
Develop an approximate solution for th rate of heat transfer through the wall.

3rd Lucture

Radial Conduction in Bodies

1- One Dimensione Radial Conduction through a Cylinder.

Assume no heat sources within the wall of the tube. If $T_1>T_2$, heat will flow outward, radially, from the inside radius, R_1, to the outside radius, R_2. The process will be described by the Fourier Law.



The differential equation governing heat diffusion is: $\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)=0$

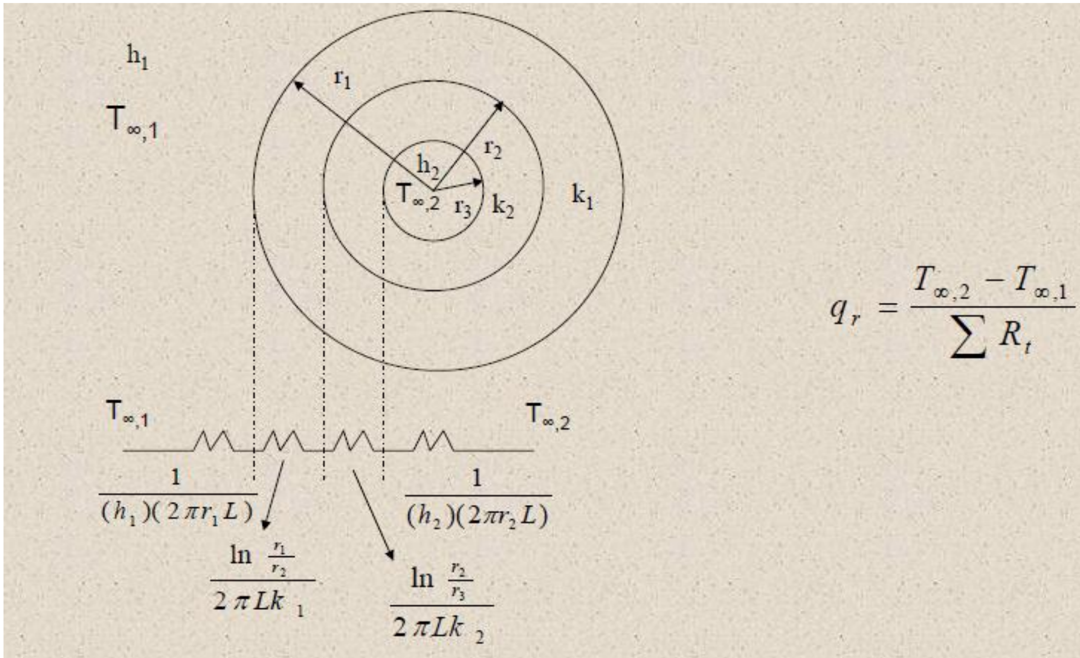
With constant k, the solution is

The heat flow rate across the wall is given by:

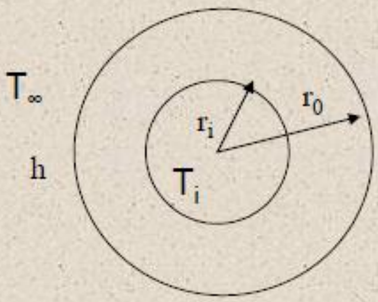
$$q_x=-kA\frac{dT}{dx}=\frac{kA}{L}(T_{s,1}-T_{s,2})=\frac{T_{s,1}-T_{s,2}}{L/kA}$$

Hence, the thermal resistance in this case can be expressed as: $\frac{\ln\frac{r_1}{r_2}}{2\pi kL}$

2- One Dimensional Radial Conduction in a Composite Cylinder



3- Critical Insulation Thickness.



Insulation Thickness : $r_o - r_i$

$$R_{tot} = \frac{\ln(\frac{r_0}{r_i})}{2\pi kL} + \frac{1}{(2\pi r_0 L)h}$$

Objective : decrease q , increases R_{tot}

Vary r_0 ; as r_0 increases ,first term increases, second term decreases.

Maximum – Minimum problem

Set $\frac{dR_{tot}}{dr_0} = 0$

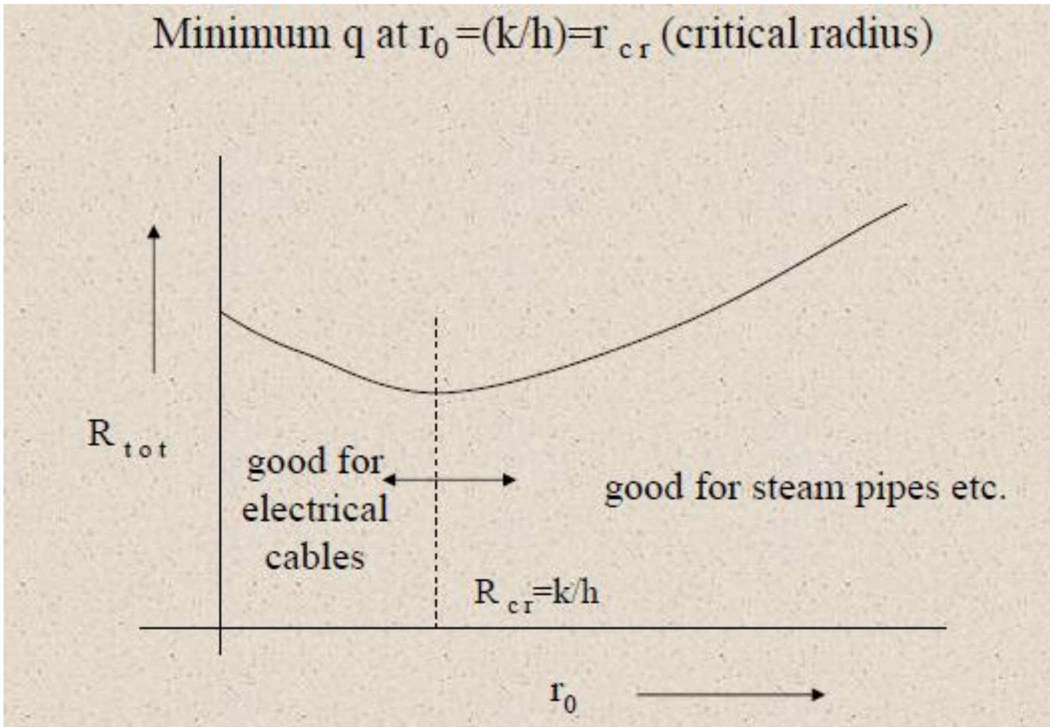
$$\frac{1}{2\pi k r_0 L} - \frac{1}{2\pi h L r_0^2} = 0$$

$$r_0 = \frac{k}{h}$$

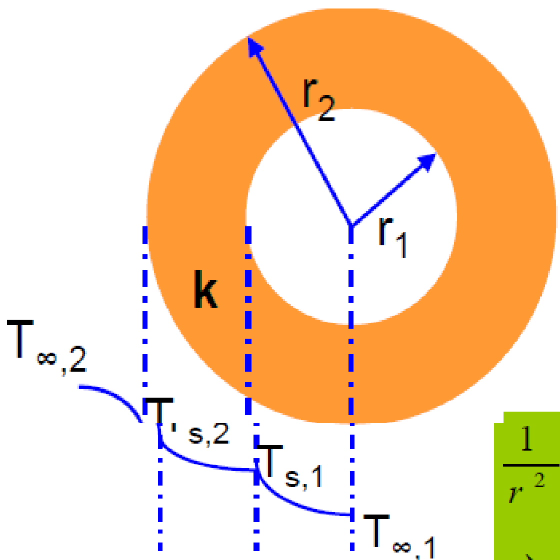
Max or Min. ? Take : $\frac{d^2 R_{tot}}{dr_0^2} = 0$ at $r_0 = \frac{k}{h}$

$$\frac{d^2 R_{tot}}{dr_0^2} = \frac{-1}{2\pi k r_0^2 L} + \frac{1}{\pi r_0^2 hL}\Bigg|_{r_0 = \frac{k}{h}}$$

$$= \frac{h^2}{2\pi L k^3} \rangle 0$$



4- One-Dimension Conduction in Sphere.



Inside Solid:

$$\frac{1}{r^2}\frac{d}{dr}\left(kr^2\frac{dT}{dr}\right) = 0$$

$$\rightarrow T(r) = T_{s,1} - \{T_{s,1} - T_{s,2}\}\left[\frac{1-(r_1/r)}{1-(r_1/r_2)}\right]$$

$$\rightarrow q_r = -kA\frac{dT}{dr} = \frac{4\pi k\left(T_{s,1} - T_{s,2}\right)}{\left(1/r_1 - 1/r_2\right)}$$

$$\rightarrow R_{t,cond} = \frac{1/r_1 - 1/r_2}{4\pi k}$$

5-Conduction with Thermal Energy Generation.

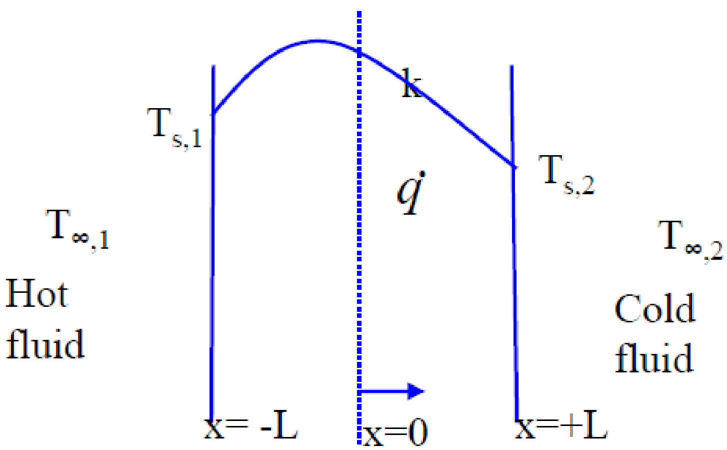
$$\dot{q} = \frac{\dot{E}}{V}$$ = Energy generation per unit volume

Applications: * current carrying conductors

* chemically reacting systems

* nuclear reactors

The Plane Wall :



Assumptions:

1D, steady state, constant k,

uniform $\dot{q}$

$$\frac{d^2 T}{dx^2} + \frac{\dot{q}}{k} = 0$$

$$Boundary\ cond.: \quad x = -L, \quad T = T_{s,1}$$

$$x = +L, \quad T = T_{s,2}$$

$$Solution: \quad T = -\frac{\dot{q}}{2k} x^2 + C_1 x + C_2$$

Use boundary conditions to find C_1 *and* C_2

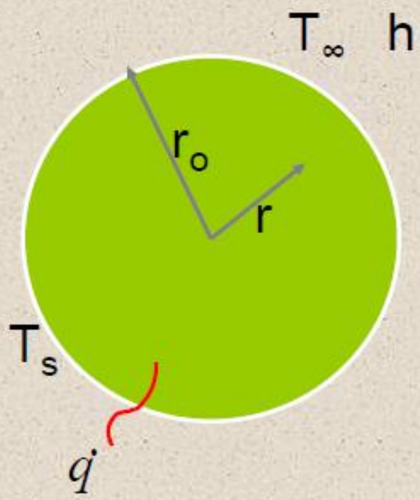
$$Final\ solution: T = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + \frac{T_{s,2} - T_{s,1}}{2}\frac{x}{L} + \frac{T_{s,2} + T_{s,1}}{2}$$

No more linear

$$Heat\ flux: \quad q''_x = -k\frac{dT}{dx}$$

Derive the expression and show that it is no more independent of x

6- Cylinder with Heat Source.



Assumptions:

1D, steady state, constant k, uniform $\dot{q}$

Start with 1D heat equation in cylindrical co-ordinates:

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{\dot{q}}{k}=0$$

$$Boundary\ cond.:\quad r=r_0,\quad T=T_s$$

$$r=0,\quad \frac{dT}{dr}=0$$

$$Solution:\quad T(r)=\frac{\dot{q}}{4k}r_0^{2}\left(1-\frac{r^2}{r_0^{2}}\right)+T_s$$

Example:

A current of 200A is passed through a stainless steel wire having a thermal conductivity K=19W/mK, diameter 3mm, and electrical resistivity R = 0.99 Ω. The length of the wire is 1m. The wire is submerged in a liquid at 110°C, and the heat transfer coefficient is 4W/m^{2}K. Calculate the centre temperature of the wire at steady state condition.

Lecture Four

Extended Surface Heat Transfer

1- Extended Surface- Fins.

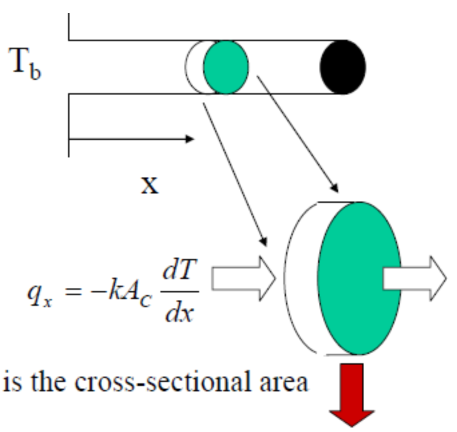
Convection: Heat transfer between a solid surface and a moving fluid is governed by the Newton's cooling law: $q = hA(T_s - T_\infty)$. Therefore, to increase the convective heat transfer, one can

- ❑ Increase the temperature difference $(T_s - T_\infty)$ between the surface and the fluid.

- ❑ Increase the convection coefficient h. This can be accomplished by increasing the fluid flow over the surface since h is a function of the flow velocity and the higher the velocity, the higher the h. Example: a cooling fan.

- ❑ Increase the contact surface area A. Example: a heat sink with fins.

2- Extended Surface Analysis.



P: the fin perimeter

A_c: the fin cross-sectional area

$$q_x = -kA_C \frac{dT}{dx} \qquad q_{x+dx} = q_x + \frac{dq_x}{dx}dx$$

A_C is the cross-sectional area

$dq_{conv} = h(dA_S)(T - T_\infty)$, where dA_S is the surface area of the element

Energy Balance: $q_x = q_{x+dx} + dq_{conv} = q_x + \frac{dq_x}{dx}dx + hdA_S(T - T_\infty)$

$$-kA_C \frac{d^2T}{dx^2}dx + hP(T - T_\infty)dx = 0, \text{ if k, } A_C \text{ are all constants.}$$

$$\frac{d^2T}{dx^2}-\frac{hP}{kA_C}(T-T_\infty)=0,\ \text{A second - order, ordinary differential equation}$$

Define a new variable $\theta(x)=T(x)-T_\infty$, so that

$$\frac{d^2\theta}{dx^2}-m^2\theta=0,\ \text{where } m^2=\frac{hP}{kA_C},\ (D^2-m^2)\theta=0$$

Characteristics equation with two real roots: $+\text{m}$ & $-\text{m}$

The general solution is of the form

$$\theta(x)=C_1e^{mx}+C_2e^{-mx}$$

To evaluate the two constants C_1 and C_2, we need to specify two boundary conditions:

The first one is obvious: the base temperature is known as $T(0)=T_b$

The second condition will depend on the end condition of the tip

For example: assume the tip is insulated and no heat transfer $d\theta/dx(x=L)=0$

The temperature distribution is given by

$$\frac{T(x)-T_\infty}{T_b-T_\infty}=\frac{\theta}{\theta_b}=\frac{\cosh m(L-x)}{\cosh mL}$$

The fin heat transfer rate is

$$q_f=-kA_C\frac{dT}{dx}(x=0)=\sqrt{hPkA_C}\tanh mL=M\tanh mL$$

These results and other solutions using different end conditions are Tabulated in HT Textbooks

3- Temperature Distribution for Fins.

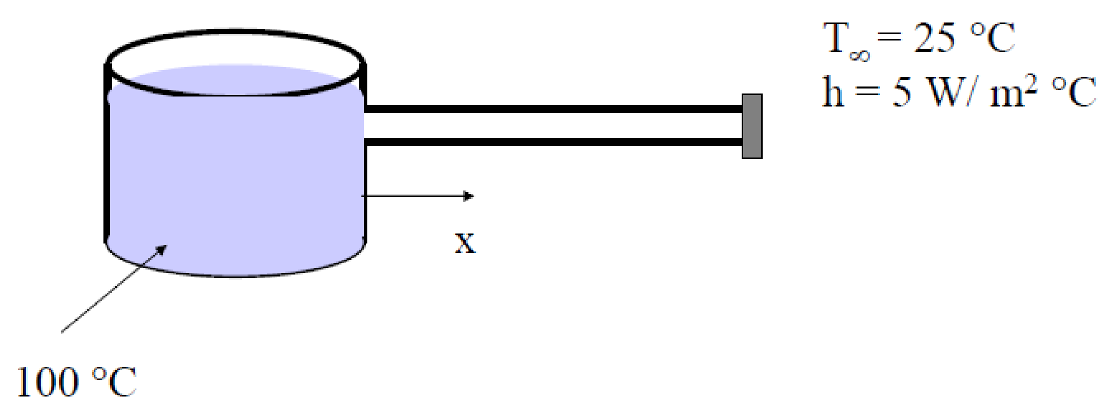
Case	Tip Condition	Temp. Distribution	Fin heat transfer
A	Convection heat transfer: $h\theta(L)=-k(d\theta/dx)_{x=L}$	$\frac{\cosh m(L-x)+(h/mk)\sinh m(L-x)}{\cosh mL+(h/mk)\sinh mL}$	$M\theta_o \frac{\sinh mL+(h/mk)\cosh mL}{\cosh mL+(h/mk)\sinh mL}$
B	Adiabatic $(d\theta/dx)_{x=L}=0$	$\frac{\cosh m(L-x)}{\cosh mL}$	$M\theta_0 \tanh mL$
C	Given temperature: $\theta(L)=\theta_L$	$\frac{(\theta_L/\theta_b)\sinh m(L-x)+\sinh m(L-x)}{\sinh mL}$	$M\theta_0 \frac{(\cosh mL-\theta_L/\theta_b)}{\sinh mL}$
D	Infinitely long fin $\theta(L)=0$	e^{-mx}	$M\ \theta_0$

$$\theta \equiv T-T_\infty, \quad m^2 \equiv \frac{hP}{kA_C}$$

$$\theta_b=\theta(0)=T_b-T_\infty, \quad M=\sqrt{hPkA_C}\,\theta_b$$

Example

❑An Aluminum pot is used to boil water as shown below. The handle of the pot is 20-cm long, 3-cm wide, and 0.5-cm thick. The pot is exposed to room air at 25°C, and the convection coefficient is 5 W/m² °C. Question: can you touch the handle when the water is boiling? (k for aluminum is 237 W/m °C)



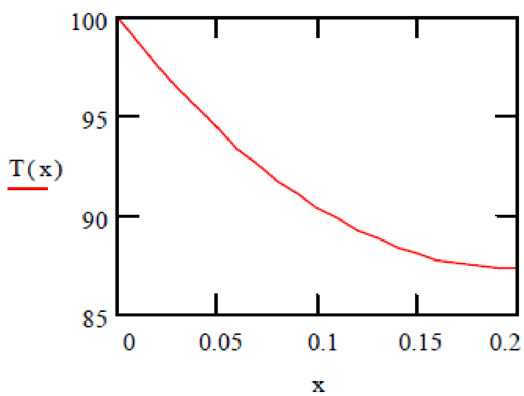
We can model the pot handle as an extended surface. Assume that there is no heat transfer at the free end of the handle. The condition matches that specified in the fins Table, case B.
$h=5\ W/m^2\ °C$, $P=2W+2t=2(0.03+0.005)=0.07(m)$, $k=237\ W/m\ °C$, $A_C=Wt=0.00015(m^2)$, $L=0.2(m)$
Therefore, $m=(hP/kA_C)^{1/2}=3.138$,
$M=\sqrt{(hPkA_C)}(T_b-T_\infty)=0.111\theta_b=0.111(100-25)=8.325(W)$

$$\frac{T(x)-T_\infty}{T_b-T_\infty}=\frac{\theta}{\theta_b}=\frac{\cosh m(L-x)}{\cosh mL}$$

$$\frac{T-25}{100-25}=\frac{\cosh[3.138(0.2-x)]}{\cosh(3.138*0.2)},$$

$$T(x)=25+62.32*\cosh[3.138(0.2-x)]$$

Plot the temperature distribution along the pot handle



As shown, temperature drops off very quickly. At the midpoint T(0.1)=90.4°C. At the end T(0.2)=87.3°C.
Therefore, it should not be safe to touch the end of the handle

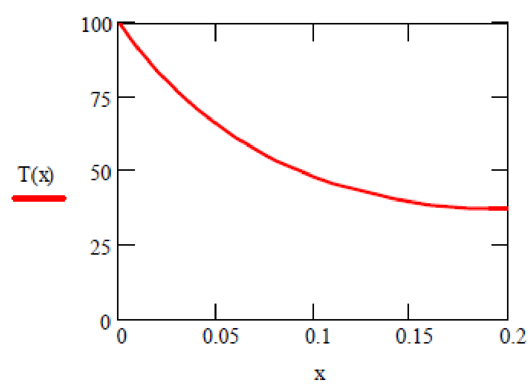
The total heat transfer through the handle can be calculated also. q_f=Mtanh(mL)=8.325*tanh(3.138*0.2)=4.632(W) Very small amount: latent heat of evaporation for water: 2257 kJ/kg. Therefore, the amount of heat loss is just enough to vaporize 0.007 kg of water in one hour.

If a stainless steel handle is used instead, what will happen: For a stainless steel, the thermal conductivity k=15 W/m°C. Use the same parameter as before:

$$m=\left(\frac{hP}{kA_C}\right)^{1/2}=12.47,\quad M=\sqrt{hPkA_C}=0.0281$$

$$\frac{T(x)-T_\infty}{T_b-T_\infty}=\frac{\cosh m(L-x)}{\cosh mL}$$

$$T(x)=25+12.3\cosh[12.47(L-x)]$$



Temperature at the handle (x=0.2 m) is only 37.3 °C, not hot at all. This example illustrates the important role played by the thermal conductivity of the material in terms of conductive heat transfer.

❑If the pot from previous lecture is made of other materials other than the aluminum, what will be the temperature distribution? Try stainless steel (k=15 W/m.K) and copper (385 W/m.K).

Recall: $h=5W/m^2°C$, $P=2W+2t=2(0.03+0.005)=0.07(m)$

$A_C=Wt=0.00015(m^2)$, $L=0.2(m)$

Therefore, $m_{ss}=(hP/kA_C)^{1/2}=12.47$, $m_{cu}=2.46$

$M_{ss}=\sqrt{(hPk_{ss}A_C)}\,(T_b-T_\infty)=0.028(100-25)=2.1(W)$

$M_{cu}=\sqrt{(hPk_{ss}A_C)}\,\theta_b=0.142(100-25)=10.66(W)$

For stainless steel, $$\frac{T_{ss}(x)-T_\infty}{T_b-T_\infty}=\frac{\theta}{\theta_b}=\frac{\cosh m(L-x)}{\cosh mL}$$

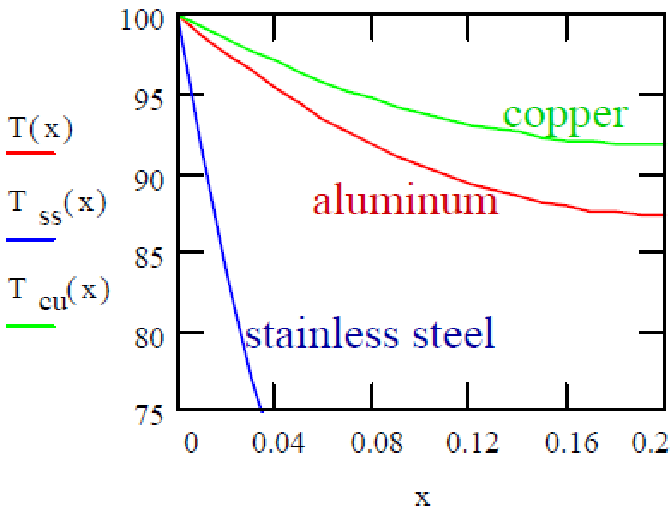
$$\frac{T_{ss}-25}{100-25}=\frac{\cosh[12.47(0.2-x)]}{\cosh(12.47*0.2)},$$

$$T_{ss}(x)=25+12.3*\cosh[12.47(0.2-x)]$$

For copper, $$\frac{T_{cu}(x)-T_\infty}{T_b-T_\infty}=\frac{\theta}{\theta_b}=\frac{\cosh m(L-x)}{\cosh mL}$$

$$\frac{T_{cu}-25}{100-25}=\frac{\cosh[2.46(0.2-x)]}{\cosh(2.46*0.2)},$$

$$T_{cu}(x)=25+66.76*\cosh[2.46(0.2-x)]$$



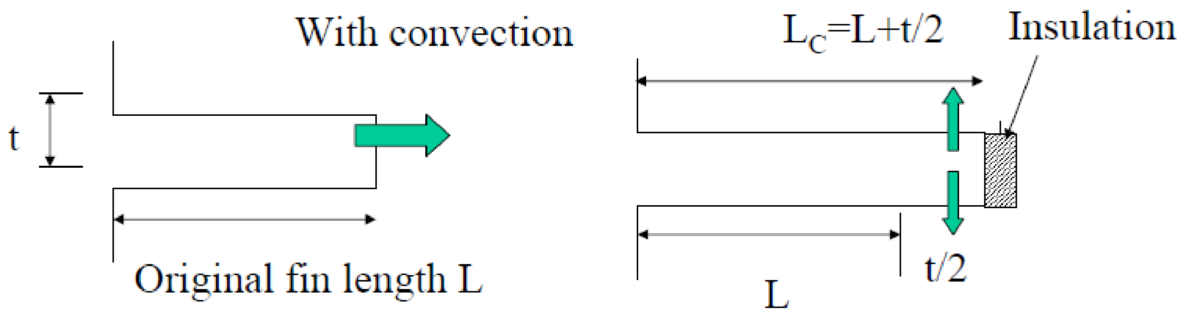
- Inside the handle of the stainless steel pot, temperature drops quickly. Temperature at the end of the handle is 37.3°C. This is because the stainless steel has low thermal conductivity and heat can not penetrate easily into the handle.
- Copper has the highest k and, correspondingly, the temperature inside the copper handle distributes more uniformly. Heat easily transfers into the copper handle.
- Question? Which material is most suitable to be used in a heat sink?

Lecture Five

Fin Specifications and Design

1- Correction Length.

- In some situations, it might be necessary to include the convective heat transfer at the tip. However, one would like to avoid using the long equation as described in case A, fins table. The alternative is to use case B instead and accounts for the convective heat transfer at the tip by extending the fin length L to $L_C=L+(t/2)$.



Then apply the adiabatic condition at the tip of the extended fin as shown above.

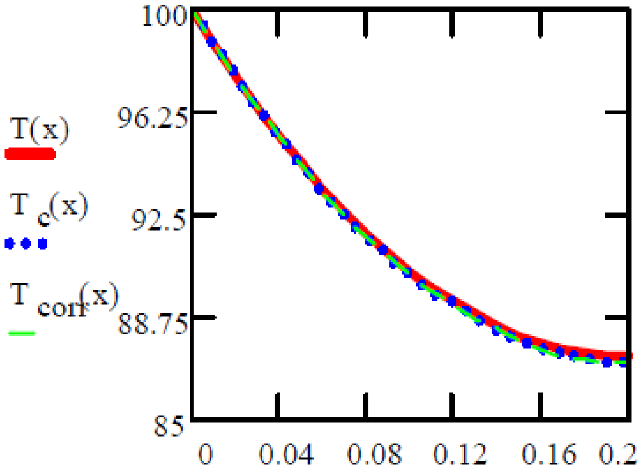
Use the same example: aluminum pot handle, *m*=3.138, the length will need to be corrected to

L_C=l+(t/2)=0.2+0.0025=0.2025(m)

$$\frac{T_{corr}(x) - T_\infty}{T_b - T_\infty} = \frac{\theta}{\theta_b} = \frac{\cosh m(L_c - x)}{\cosh mL_c}$$

$$\frac{T_{corr} - 25}{100 - 25} = \frac{\cosh[3.138(0.2025 - x)]}{\cosh(3.138 * 0.2025)},$$

$$T_{corr}(x) = 25 + 62.05 * \cosh[3.138(0.2025 - x)]$$



$T(0.2)=87.32$ °C
$T_c(0.2)=87.09$ °C
$T_{corr}(0.2025)=87.05$ °C

slight improvement over the uncorrected solution

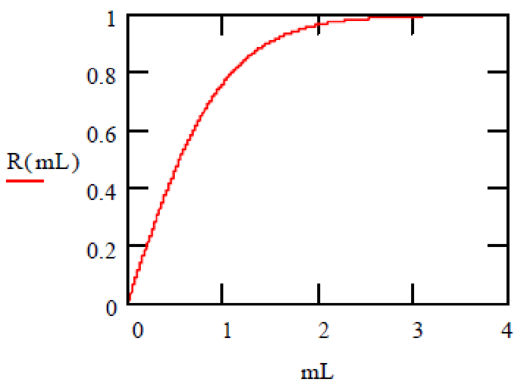
❑ The correction length can be determined by using the formula: $L_c=L+(A_c/P)$, where A_c is the cross-sectional area and P is the perimeter of the fin at the tip.

❑ Thin rectangular fin: $A_c=Wt$, $P=2(W+t)\approx 2W$, since $t << W$
$L_c=L+(A_c/P)=L+(Wt/2W)=L+(t/2)$

❑ Cylindrical fin: $A_c=(\pi/4)D^2$, $P=\pi D$, $L_c=L+(A_c/P)=L+(D/4)$

❑ Square fin: $A_c=W^2$, $P=4W$,
$L_c=L+(A_c/P)=L+(W^2/4W)=L+(W/4)$

❑ In general, the longer the fin, the higher the heat transfer. However, a long fin means more material and increased size and cost. Question: how do we determine the optimal fin length? Use the rectangular fin as an example:



$q_f = M \tanh mL$, for an adiabatic tip fin

$(q_f)_\infty = M$, for an infinitely long fin

Their ratio: $R(mL) = \dfrac{q_f}{(q_f)_\infty} = \tanh mL$

Note: heat transfer increases with mL as expected. Initially the rate of change is large and slows down drastically when mL> 2.

R(1)=0.762, means any increase beyond mL=1 will increase no more than 23.8% of the fin heat transfer.

2- <u>Temperature Distribution.</u>

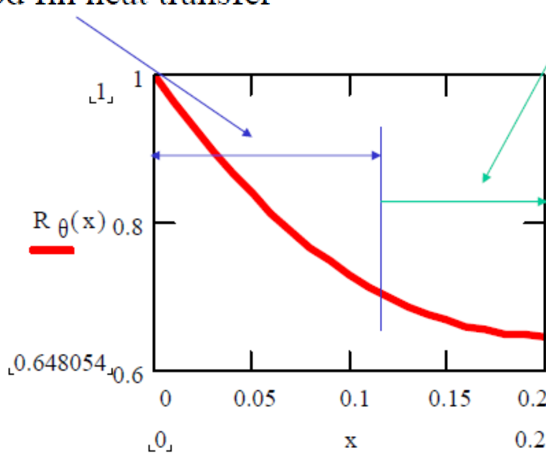
For an adiabatic tip fin case:

$$R_\theta = \frac{T - T_\infty}{T_b - T_\infty} = \frac{\cosh m(L-x)}{\cosh mL}$$

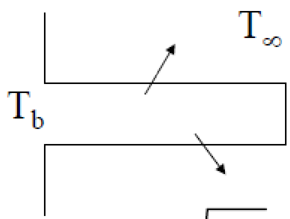
➢ Use m=5, and L=0.2 as an example:

High ΔT, good fin heat transfer

Low ΔT, poor fin heat transfer



3- Fin Design.



Total heat loss: q_f=Mtanh(mL) for an adiabatic fin, or q_f=Mtanh(mL_C) if there is convective heat transfer at the tip

where $m=\sqrt{\frac{hP}{kA_c}}$, and $M=\sqrt{hPkA_C}\,\theta_b=\sqrt{hPkA_C}\,(T_b-T_\infty)$

Use the thermal resistance concept:

$$q_f=\sqrt{hPkA_C}\,\tanh(mL)(T_b-T_\infty)=\frac{(T_b-T_\infty)}{R_{t,f}}$$

where $R_{t,f}$ is the thermal resistance of the fin.

For a fin with an adiabatic tip, the fin resistance can be expressed as

$$R_{t,f}=\frac{(T_b-T_\infty)}{q_f}=\frac{1}{\sqrt{hPkA_C}\,[\tanh(mL)]}$$

4- Fin Effectiveness.

fin effectiveness ε_f: Ratio of fin heat transfer and the heat transfer without the fin. For an adiabatic fin:

$$\varepsilon_f=\frac{q_f}{q}=\frac{q_f}{hA_C(T_b-T_\infty)}=\frac{\sqrt{hPkA_C}\,\tanh(mL)}{hA_C}=\sqrt{\frac{kP}{hA_C}}\tanh(mL)$$

If the fin is long enough, mL>2, tanh(mL) $\rightarrow$ 1,

it can be considered an infinite fin (case D of table3.4)

$$\varepsilon_f\rightarrow\sqrt{\frac{kP}{hA_C}}=\sqrt{\frac{k}{h}\left(\frac{P}{A_C}\right)}$$

In order to enhance heat transfer, $\varepsilon_f>1$.

However, $\varepsilon_f\geq 2$ will be considered justifiable

If ε_f<1 then we have an insulator instead of a heat fin

$$\varepsilon_f \rightarrow \sqrt{\frac{kP}{hA_C}} = \sqrt{\frac{k}{h}\left(\frac{P}{A_C}\right)}$$

- To increase ε_f, the fin's material should have higher thermal conductivity, k.
- It seems to be counterintuitive that the lower convection coefficient, h, the higher ε_f. But it is not because if h is very high, it is not necessary to enhance heat transfer by adding heat fins. Therefore, heat fins are more effective if h is low. Observation: If fins are to be used on surfaces separating gas and liquid. Fins are usually placed on the gas side. (Why?)
- P/AC should be as high as possible. Use a square fin with a dimension of W by W as an example: P=4W, AC=W2, P/AC=(4/W). The smaller W, the higher the P/AC, and the higher εf.
- Conclusion: It is preferred to use thin and closely spaced (to increase the total number) fins.

The effectiveness of a fin can also be characterized as

$$\varepsilon_f = \frac{q_f}{q} = \frac{q_f}{hA_C(T_b - T_\infty)} = \frac{(T_b - T_\infty)/R_{t,f}}{(T_b - T_\infty)/R_{t,h}} = \frac{R_{t,h}}{R_{t,f}}$$

It is a ratio of the thermal resistance due to convection to the thermal resistance of a fin. In order to enhance heat transfer, the fin's resistance should be lower than that of the resistance due only to convection.

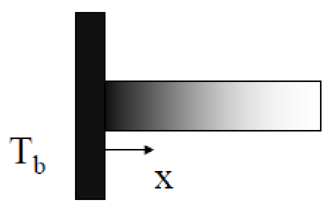
5- Fin Efficiency.

Define Fin efficiency: $\eta_f = \dfrac{q_f}{q_{max}}$

where q_{max} represents an idealized situation such that the fin is made up of material with infinite thermal conductivity. Therefore, the fin should be at the same temperature as the temperature of the base.

$q_{max} = hA_f(T_b - T_\infty)$

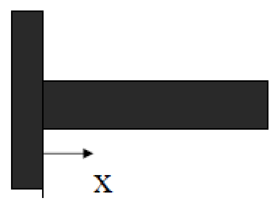
T(x)<T_b for heat transfer to take place



Total fin heat transfer q_f

Real situation

For infinite k
T(x)=T_b, the heat transfer is maximum



Ideal heat transfer q_{max}

Ideal situation

Use an adiabatic rectangular fin as an example:

$$\eta_f = \frac{q_f}{q_{max}} = \frac{M \tanh mL}{hA_f(T_b - T_\infty)} = \frac{\sqrt{hPkA_c}(T_b - T_\infty)\tanh mL}{hPL(T_b - T_\infty)}$$

$$= \frac{\tanh mL}{\sqrt{\frac{hP}{kA_c}}L} = \frac{\tanh mL}{mL} \quad \text{(see Table 3.5 for } \eta_f \text{ of common fins)}$$

The fin heat transfer: $q_f = \eta_f q_{max} = \eta_f hA_f(T_b - T_\infty)$

$$q_f = \frac{T_b - T_\infty}{1/(\eta_f hA_f)} = \frac{T_b - T_\infty}{R_{t,f}}, \text{ where } R_{t,f} = \frac{1}{\eta_f hA_f}$$

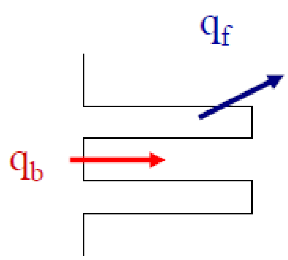
Thermal resistance for a single fin.

As compared to convective heat transfer: $R_{t,b} = \dfrac{1}{hA_b}$

In order to have a lower resistance as that is required to enhance heat transfer: $R_{t,b} > R_{t,f}$ or $A_b < \eta_f A_f$

6- Overall Fin Efficiency.

Overall fin efficiency for an array of fins:



Define terms: A_b: base area exposed to coolant

A_f: surface area of a single fin

A_t: total area including base area and total finned surface, $A_t = A_b + NA_f$

N: total number of fins

$$q_t = q_b + Nq_f = hA_b(T_b - T_\infty) + N\eta_f hA_f(T_b - T_\infty)$$

$$= h[(A_t - NA_f) + N\eta_f A_f](T_b - T_\infty) = h[A_t - NA_f(1-\eta_f)](T_b - T_\infty)$$

$$= hA_t[1 - \frac{NA_f}{A_t}(1-\eta_f)](T_b - T_\infty) = \eta_O hA_t(T_b - T_\infty)$$

Define overall fin efficiency: $\eta_O = 1 - \frac{NA_f}{A_t}(1-\eta_f)$

7- Heat Transfer from a Fin Array.

$$q_t = hA_t\eta_O(T_b - T_\infty) = \frac{T_b - T_\infty}{R_{t,O}} \quad \text{where } R_{t,O} = \frac{1}{hA_t\eta_O}$$

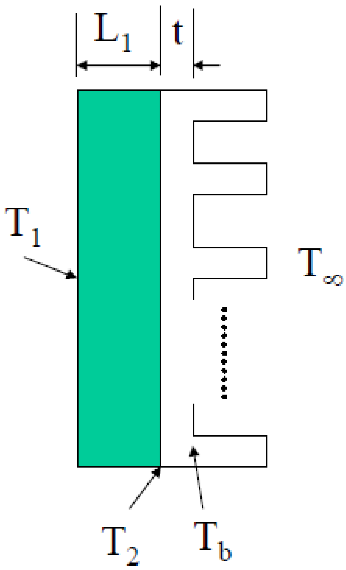
Compare to heat transfer without fins

$$q = hA(T_b - T_\infty) = h(A_b + NA_{b,f})(T_b - T_\infty) = \frac{1}{hA}$$

where $A_{b,f}$ is the base area (unexposed) for the fin

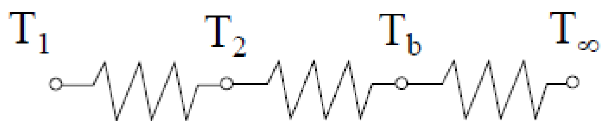
To enhance heat transfer $A_t\eta_O >> A$

That is, to increase the effective area $\eta_O A_t$.



$$A = A_b + NA_{b,f}$$

$$R_b = t/(k_b A)$$



$$R_1 = L_1/(k_1 A) \qquad R_{t,o} = 1/(hA_t\eta_o)$$

$$q = \frac{T_1 - T_\infty}{\sum R} = \frac{T_1 - T_\infty}{R_1 + R_b + R_{t,o}}$$

Lecture – Six

Multi-Dimensional Steady State Heat Conduction

1- Introduction.

Heat Diffusion Equation

$$\rho c_p \frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \dot{q} = k\nabla^2 T + \dot{q}$$

- ❑ This equation governs the Cartesian, temperature distribution for a three-dimensional unsteady, heat transfer problem involving heat generation.
- ❑ For steady state $\partial / \partial t = 0$
- ❑ No generation $\dot{q} = 0$
- ❑ To solve for the full equation, it requires a total of six boundary conditions: two for each direction. Only one initial condition is needed to account for the transient behavior.

2- Two Dimension Steady State Case.

For a 2 - D, steady state situation, the heat equation is simplified to

$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$, it needs two boundary conditions in each direction.

There are three approaches to solve this equation:

- ❑ Numerical Method: Finite difference or finite element schemes, usually will be solved using computers.
- ❑ Graphical Method: Limited use. However, the conduction shape factor concept derived under this concept can be useful for specific configurations. (see Table 4.1 for selected configurations)
- ❑ Analytical Method: The mathematical equation can be solved using techniques like the method of separation of variables. (refer to handout)

3- Conduction Shape Factor.

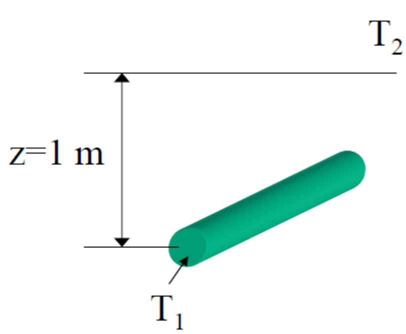
❑ This approach applied to 2-D conduction involving two isothermal surfaces, with all other surfaces being adiabatic. The heat transfer from one surface (at a temperature T_1) to the other surface (at T_2) can be expressed as: $q=Sk(T_1-T_2)$ where k is the thermal conductivity of the solid and S is the conduction shape factor.

❑ The shape factor can be related to the thermal resistance: $q=Sk(T_1-T_2)=(T_1-T_2)/(1/kS)= (T_1-T_2)/R_t$

where $R_t = 1/(kS)$

❑ 1-D heat transfer can use shape factor also. Ex: Heat transfer inside a plane wall of thickness L is $q=kA(\Delta T/L)$, $S=A/L$

❑ An Alaska oil pipe line is buried in the earth at a depth of 1 m. The horizontal pipe is a thin-walled of outside diameter of 50 cm. The pipe is very long and the averaged temperature of the oil is 100°C and the ground soil temperature is at -20 °C (k_{soil}=0.5W/m.K), estimate the heat loss per unit length of pipe.



From Table 8.7, case 1.
L>>D, z>3D/2

$$S = \frac{2\pi L}{\ln(4z/D)} = \frac{2\pi(1)}{\ln(4/0.5)} = 3.02$$

$$q = kS(T_1 - T_2) = (0.5)(3.02)(100+20)$$

$= 181.2(W)$ heat loss for every meter of pipe

If the mass flow rate of the oil is 2 kg/s and the specific heat of the oil is 2 kJ/kg.K, determine the temperature change in 1 m of pipe length.

$$q = \dot{m}C_P\Delta T,\ \Delta T = \frac{q}{\dot{m}C_P} = \frac{181.2}{2000*2} = 0.045(°C)$$

Therefore, the total temperature variation can be significant if the pipe is very long. For example, 45°C for every 1 km of pipe length.

- ❑ Heating might be needed to prevent the oil from freezing up.
- ❑ The heat transfer can not be considered constant for a long pipe

Heat Transfer at section with a temperature T(x)

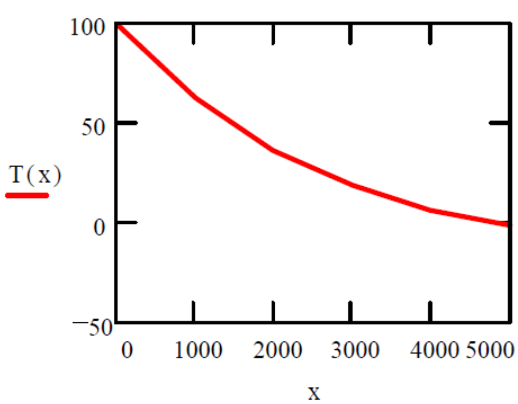
$$q = \frac{2\pi k(dx)}{\ln(4z/D)}(T+20) = 1.51(T+20)(dx)$$

Energy balance: $\dot{m}C_P T - q = \dot{m}C_P(T+dT)$

$$\dot{m}C_P\frac{dT}{dx} + 1.51(T+20) = 0,\ \frac{dT}{T+20} = -0.000378dx, \text{integrate}$$

$$T(x) = -20 + Ce^{-0.000378x},\ \text{at inlet x = 0, T(0) = 100°C, C = 120}$$

$$T(x) = -20 + 120e^{-0.000378x}$$



- ❑ Temperature drops exponentially from the initial temp. of 100°C
- ❑ It reaches 0°C at x=4740 m, therefore, reheating is required every 4.7 km.

4- Numerical Methods.

❑ Due to the increasing complexities encountered in the development of modern technology, analytical solutions usually are not available. For these problems, numerical solutions obtained using high-speed computer are very useful, especially when the geometry of the object of interest is irregular, or the boundary conditions are nonlinear. In numerical analysis, two different approaches are commonly used: The finite difference and the finite element methods. In heat transfer problems, the finite difference method is used more often and will be discussed here.

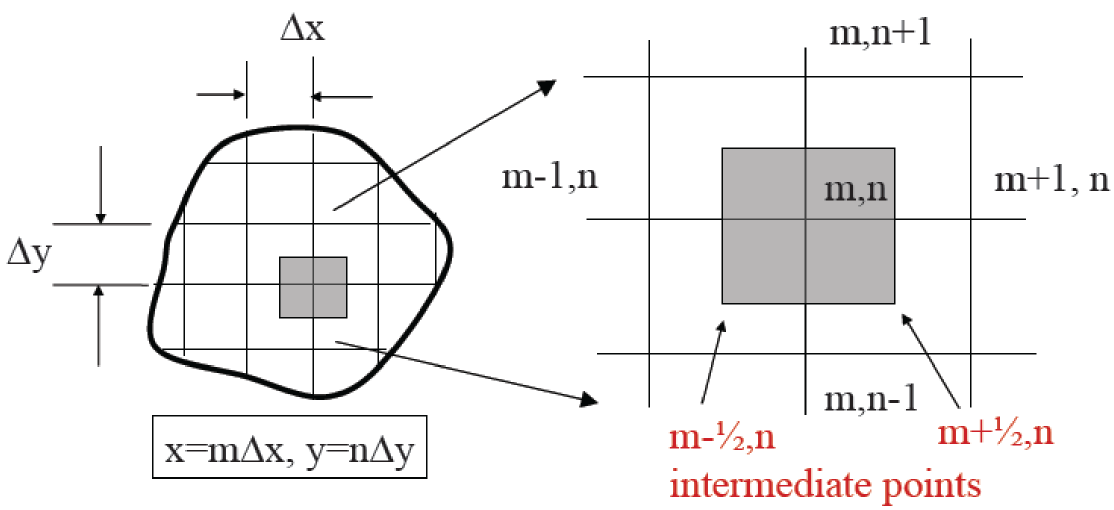
❑ The finite difference method involves:

- Establish nodal networks
- Derive finite difference approximations for the governing equation at both interior and exterior nodal points
- Develop a system of simultaneous algebraic nodal equations
- Solve the system of equations using numerical schemes

❑ The basic idea is to subdivide the area of interest into sub-volumes with the distance between adjacent nodes by Δx and Δy as shown. If the distance between points is small enough, the differential equation can be approximated locally by a set of finite difference equations. Each node now represents a small region where the nodal temperature is a measure of the average temperature of the region.

5- Finite Difference Approximation.

Example



Heat Diffusion Equation: $\nabla^2 T + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$,

where $\alpha = \frac{k}{\rho C_P V}$ is the thermal diffusivity

No generation and steady state: $\dot{q}=0$ and $\frac{\partial}{\partial t} = 0, \Rightarrow \nabla^2 T = 0$

First, approximated the first order differentiation at intermediate points (m+1/2,n) & (m-1/2,n)

$$\left.\frac{\partial T}{\partial x}\right|_{(m+1/2,n)} \approx \left.\frac{\Delta T}{\Delta x}\right|_{(m+1/2,n)} = \frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$

$$\left.\frac{\partial T}{\partial x}\right|_{(m-1/2,n)} \approx \left.\frac{\Delta T}{\Delta x}\right|_{(m-1/2,n)} = \frac{T_{m,n} - T_{m-1,n}}{\Delta x}$$

Next, approximate the second order differentiation at m,n

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{m,n} \approx \frac{\left.\partial T/\partial x\right|_{m+1/2,n} - \left.\partial T/\partial x\right|_{m-1/2,n}}{\Delta x}$$

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{m,n} \approx \frac{T_{m+1,n} + T_{m-1,n} - 2T_{m,n}}{(\Delta x)^2}$$

Similarly, the approximation can be applied to the other dimension y

$$\left.\frac{\partial^2 T}{\partial y^2}\right|_{m,n} \approx \frac{T_{m,n+1} + T_{m,n-1} - 2T_{m,n}}{(\Delta y)^2}$$

$$\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)_{m,n} \approx \frac{T_{m+1,n} + T_{m-1,n} - 2T_{m,n}}{(\Delta x)^2} + \frac{T_{m,n+1} + T_{m,n-1} - 2T_{m,n}}{(\Delta y)^2}$$

To model the steady state, no generation heat equation: $\nabla^2 T = 0$

This approximation can be simplified by specify $\Delta x = \Delta y$

and the nodal equation can be obtained as

$$T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1} - 4T_{m,n} = 0$$

This equation approximates the nodal temperature distribution based on the heat equation. This approximation is improved when the distance between the adjacent nodal points is decreased:

Since $\lim(\Delta x \to 0)\frac{\Delta T}{\Delta x} = \frac{\partial T}{\partial x}, \lim(\Delta y \to 0)\frac{\Delta T}{\Delta y} = \frac{\partial T}{\partial y}$

6- A System of Algebraic Equations.

❑ The nodal equations derived previously are valid for all interior points satisfying the steady state, no generation heat equation. For each node, there is one such equation.

For example: for nodal point m=3, n=4, the equation is

$T_{2,4} + T_{4,4} + T_{3,3} + T_{3,5} - 4T_{3,4} = 0$

$T_{3,4} = (1/4)(T_{2,4} + T_{4,4} + T_{3,3} + T_{3,5})$

❑ Nodal relation table for exterior nodes (boundary conditions) can be found in standard heat transfer textbooks (Table 4.2 of our textbook).

❑ Derive one equation for each nodal point (including both interior and exterior points) in the system of interest. The result is a system of N algebraic equations for a total of N nodal points.

Matrix Form

The system of equations:

$$\begin{aligned}
a_{11}T_1 + a_{12}T_2 + \cdots + a_{1N}T_N &= C_1 \\
a_{21}T_1 + a_{22}T_2 + \cdots + a_{2N}T_N &= C_2 \\
\vdots \qquad \vdots \qquad \vdots \quad \vdots \qquad &\quad \vdots \\
a_{N1}T_1 + a_{N2}T_2 + \cdots + a_{NN}T_N &= C_N
\end{aligned}$$

A total of N algebraic equations for the N nodal points and the system can be expressed as a matrix formulation: [**A**][**T**]=[**C**]

$$\text{where } A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix}, T = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_N \end{bmatrix}, C = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_N \end{bmatrix}$$

Lecture Seven

Numerical Technique solutions

1- Numerical Solution.

❑Matrix form: [A][T]=[C].

From linear algebra: $[A]^{-1}[A][T]=[A]^{-1}[C]$, $[T]=[A]^{-1}[C]$

where $[A]^{-1}$ is the inverse of matrix [A]. [T] is the solution vector.

❑ Matrix inversion requires cumbersome numerical computations and is not efficient if the order of the matrix is high (>10)

❑ Gauss elimination method and other matrix solvers are usually available in many numerical solution package. For example, "Numerical Recipes" by Cambridge University Press or their web source at www.nr.com.

❑ For high order matrix, iterative methods are usually more efficient. The famous Jacobi & Gauss-Seidel iteration methods will be introduced in the following.

Iteration

General algebraic equation for nodal point:

$$\sum_{j=1}^{i-1} a_{ij}T_j + a_{ii}T_i + \sum_{j=i+1}^{N} a_{ij}T_j = C_i,$$

Replace (k) by (k-1) for the Jacobi iteration

$$(\text{Example}: a_{31}T_1 + a_{32}T_2 + a_{33}T_3 + \cdots + a_{1N}T_N = C_1,\ i=3)$$

Rewrite the equation of the form:

$$T_i^{(k)} = \frac{C_i}{a_{ii}} - \sum_{j=1}^{i-1}\frac{a_{ij}}{a_{ii}}T_j^{(k)} - \sum_{j=i+1}^{N}\frac{a_{ij}}{a_{ii}}T_j^{(k-1)}$$

- (k) - specify the level of the iteration, (k-1) means the present level and (k) represents the new level.
- An initial guess (k=0) is needed to start the iteration.
- By substituting iterated values at (k-1) into the equation, the new values at iteration (k) can be estimated
- The iteration will be stopped when max| Ti(k)-Ti(k-1) |≤ε, where ε specifies a predetermined value of acceptable error

Example

Solve the following system of equations using (a) the Jacobi methods, (b) the Gauss Seidel iteration method.

$$\begin{bmatrix} 4 & 2 & 1 \\ -1 & 2 & 0 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 11 \\ 3 \\ 16 \end{bmatrix}$$

$$4X + 2Y + Z = 11,$$
$$-X + 2Y + 0 * Z = 3,$$
$$2X + Y + 4Z = 16$$

Reorganize into new form:

$$X = \frac{11}{4} - \frac{1}{2}Y - \frac{1}{4}Z$$

$$Y = \frac{3}{2} + \frac{1}{2}X + 0 * Z$$

$$Z = 4 - \frac{1}{2}X - \frac{1}{4}Y$$

- (a) Jacobi method: use initial guess X0=Y0=Z0=1,
- stop when max| Xk-Xk-1,Yk-Yk-1,Zk-Zk-1 | ≤ 0.1
- First iteration:
- X1= (11/4) - (1/2)Y0 - (1/4)Z0 = 2
- Y1= (3/2) + (1/2)X0 = 2
- Z1= 4 - (1/2) X0 - (1/4)Y0 = 13/4

Second iteration: use the iterated values $X^1=2$, $Y^1=2$, $Z^1=13/4$

$X^2 = (11/4) - (1/2)Y^1 - (1/4)Z^1 = 15/16$

$Y^2 = (3/2) + (1/2)X^1 = 5/2$

$Z^2 = 4 - (1/2) X^1 - (1/4)Y^1 = 5/2$

Converging Process:

$$[1,1,1],\ \left[2,2,\frac{13}{4}\right],\ \left[\frac{15}{16},\frac{5}{2},\frac{5}{2}\right],\ \left[\frac{7}{8},\frac{63}{32},\frac{93}{32}\right],\ \left[\frac{133}{128},\frac{31}{16},\frac{393}{128}\right]$$

$\left[\frac{519}{512},\frac{517}{256},\frac{767}{256}\right]$. Stop the iteration when

$$\max\left|X^5-X^4, Y^5-Y^4, Z^5-Z^4\right| \le 0.1$$

Final solution [1.014, 2.02, 2.996]

Exact solution [1, 2, 3]

(b) Gauss-Seidel iteration: Substitute the iterated values into the iterative process immediately after they are computed.

Use initial guess $X^0 = Y^0 = Z^0 = 1$

$$X = \frac{11}{4} - \frac{1}{2}Y - \frac{1}{4}Z,\ Y = \frac{3}{2} + \frac{1}{2}X,\ Z = 4 - \frac{1}{2}X - \frac{1}{4}Y$$

First iteration: $X^1 = \frac{11}{4} - \frac{1}{2}(Y^0) - \frac{1}{4}(Z^0) = 2$ Immediate substitution

$$Y^1 = \frac{3}{2} + \frac{1}{2}X^1 = \frac{3}{2} + \frac{1}{2}(2) = \frac{5}{2}$$

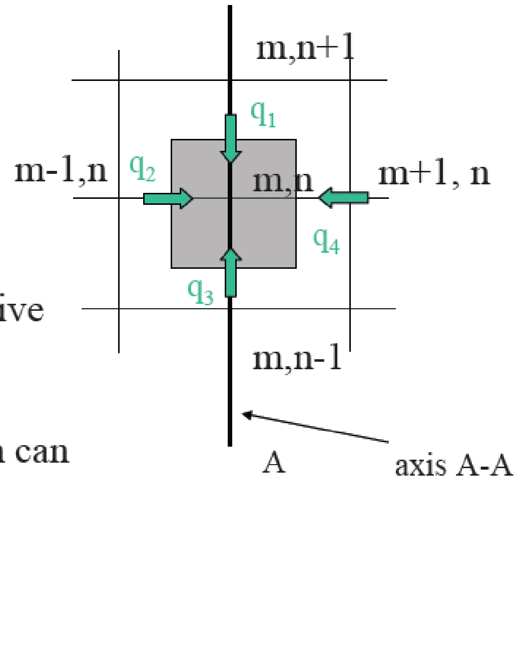
$$Z^1 = 4 - \frac{1}{2}X^1 - \frac{1}{4}Y^1 = 4 - \frac{1}{2}(2) - \frac{1}{4}\left(\frac{5}{2}\right) = \frac{19}{8}$$

Converging process: $[1,1,1],\ \left[2,\frac{5}{2},\frac{19}{8}\right],\ \left[\frac{29}{32},\frac{125}{64},\frac{783}{256}\right],\ \left[\frac{1033}{1024},\frac{4095}{2048},\frac{24541}{8192}\right]$

The iterated solution [1.009, 1.9995, 2.996] and it converges faster

2- Numerical Method (Special Cases)

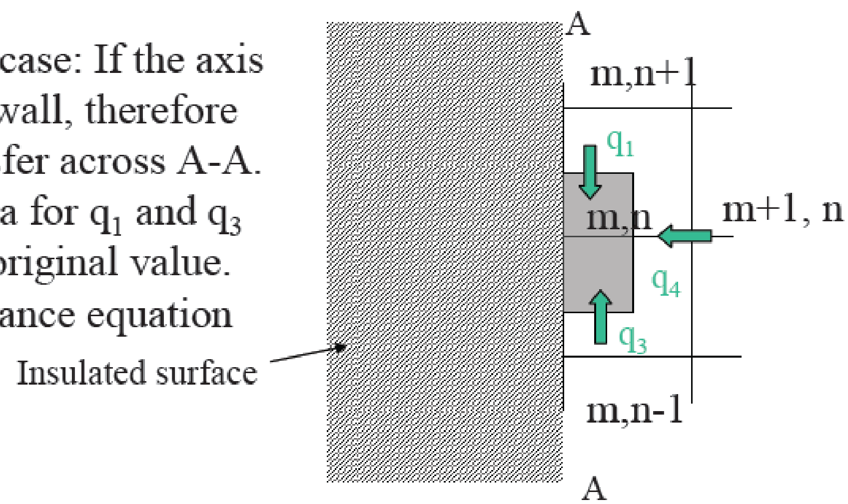
❑ For all the special cases discussed in the following, the derivation will be based on the standard nodal point configuration as shown to the right.



☐ Symmetric case: symmetrical relative to the A-A axis.
In this case, $T_{m-1,n}=T_{m+1,n}$
Therefore the standard nodal equation can be written as

$$T_{m+1,n}+T_{m-1,n}+T_{m,n+1}+T_{m,n-1}-4T_{m,n}$$
$$=2T_{m+1,n}+T_{m,n+1}+T_{m,n-1}-4T_{m,n}=0$$

☐ Insulated surface case: If the axis A-A is an insulated wall, therefore there is no heat transfer across A-A. Also, the surface area for q_1 and q_3 is only half of their original value. Write the energy balance equation ($q_2=0$):



$$q_1+q_3+q_4=0$$

$$k\left(\frac{\Delta x}{2}\right)\frac{T_{m,n+1}-T_{m,n}}{\Delta y}+k\left(\frac{\Delta x}{2}\right)\frac{T_{m,n-1}-T_{m,n}}{\Delta y}+k\Delta y\frac{T_{m+1,n}-T_{m,n}}{\Delta x}=0$$

$$2T_{m+1,n}+T_{m,n+1}+T_{m,n-1}-4T_{m,n}=0$$

This equation is identical to the symmetrical case discussed previously.

☐ With internal generation G=gV where g is the power generated per unit volume (W/m³). Based on the energy balance concept:

$$q_1 + q_2 + q_3 + q_4 + G$$

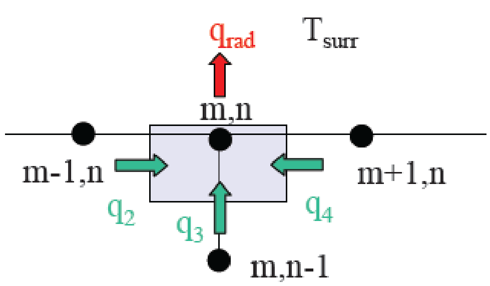
$$q_1 + q_2 + q_3 + q_4 + g(\Delta x)(\Delta y)(1) = 0$$

Use 1 to represent the dimension along the z-direction.

$$k(T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1} - 4T_{m,n}) + g\left(\Delta x\right)^2 = 0$$

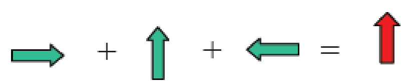
$$T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1} - 4T_{m,n} + \frac{g(\Delta x)^2}{k} = 0$$

☐ Radiation heat exchange with respect to the surrounding (assume no convection, no generation to simplify the derivation). Given surface emissivity ε, surrounding temperature T_{surr}.



From energy balance concept:

$q_2+q_3+q_4=q_{rad}$



$$k\left(\frac{\Delta y}{2}\right)\frac{T_{m-1,n} - T_{m,n}}{\Delta x} + k\left(\Delta x\right)\frac{T_{m,n-1} - T_{m,n}}{\Delta y} + k\left(\frac{\Delta y}{2}\right)\frac{T_{m+1,n} - T_{m,n}}{\Delta x} = \varepsilon\sigma\left(\Delta x\right)\left(T_{m,n}{}^4 - T_{surr}{}^4\right)$$

$$k\left(T_{m-1,n} + T_{m+1,n} + 2T_{m,n-1} - 4T_{m,n}\right) = 2\varepsilon\sigma\left(\Delta x\right)\left(T_{m,n}^4 - T_{surr}^4\right)$$

$$T_{m-1,n} + T_{m+1,n} + 2T_{m,n-1} - 4T_{m,n} - \boxed{\frac{2\varepsilon\sigma\left(\Delta x\right)}{k}T_{m,n}^4} = -2\frac{\varepsilon\sigma\left(\Delta x\right)}{k}T_{surr}^4$$

Nonlinear term, cannot solve by using iteration technique.

Lecture Eight

Unsteady State Heat Conduction

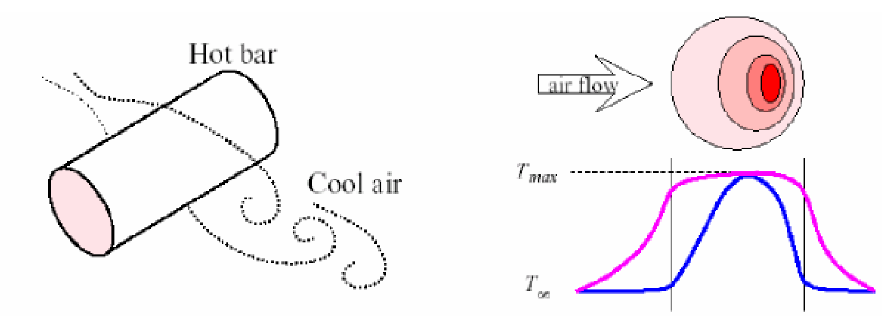
1- Unsteady Heat Transfer.

Many heat transfer problems require the understanding of the complete time history of the temperature variation. For example, in metallurgy, the heat treating process can be controlled to directly affect the characteristics of the processed materials. Annealing (slow cool) can soften metals and improve ductility. On the other hand, quenching (rapid cool) can harden the strain boundary and increase strength. In order to characterize this transient behavior, the full unsteady equation is needed:

$$\rho c \frac{\partial T}{\partial t} = k\nabla^2 T, \text{ or } \frac{1}{\alpha}\frac{\partial T}{\partial t} = \nabla^2 T$$

where $\alpha = \dfrac{k}{\rho c}$ is the thermal diffusivity

"A heated/cooled body at T_i is suddenly exposed to fluid at T_∞ with a known heat transfer coefficient . Either evaluate the temperature at a given time, or find time for a given temperature."



Q: "How good an approximation would it be to say the bar is more or less *isothermal*?"
A: "Depends on the relative importance of the thermal conductivity in the thermal circuit compared to the convective heat transfer coefficient".

2- Biot Number (Bi).

•Defined to describe the relative resistance in a thermal circuit of the convection compared

$$Bi = \frac{hL_c}{k} = \frac{L_c / kA}{1/hA} = \frac{\text{Internal conduction resistance within solid}}{\text{External convection resistance at body surface}}$$

L_c is a characteristic length of the body

Bi→0: No conduction resistance at all. The body is isothermal.

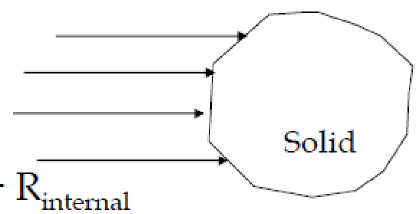
Small Bi: Conduction resistance is less important. The body may still be approximated as isothermal (purple temp. plot in figure) *Lumped capacitance analysis* can be performed.

Large Bi: Conduction resistance is significant. The body cannot be treated as isothermal (blue temp. plot in figure).

3- Lumped Parameter Analysis.

Transient heat transfer with no internal resistance.

Valid for Bi<0.1



Total Resistance= $R_{external} + R_{internal}$

GE: $$\frac{dT}{dt} = -\frac{hA}{mc_p}(T - T_\infty)$$ BC: $$T(t=0) = T_i$$

Solution: $let\ \Theta = T - T_\infty$, therefore

$$\frac{d\Theta}{dt} = -\frac{hA}{mc_p}\Theta$$

$$\Theta_i = T_i - T_\infty$$

$$\ln\frac{\Theta}{\Theta_i} = -\frac{hA}{mc_p}t$$

$$\frac{\Theta}{\Theta_i} = e^{-\frac{hA}{mc_p}t}$$

$$\frac{T - T_\infty}{T_i - T_\infty} = e^{-t \Big/ \frac{mc_p}{hA}}$$

- To determine the temperature at a given time, or
- To determine the time required for the temperature to reach a specified value.

Note: Temperature function **only of time** and **not** of space!

$$\frac{T - T_\infty}{T_0 - T_\infty} = \exp\left(-\frac{hA}{\rho c V}t\right)$$

$$\frac{hA}{\rho c V}t = \left(\frac{hL_c}{k}\right)\left(\frac{k}{\rho c}\right)\frac{1}{L_c}\frac{1}{L_c}t = Bi\frac{\alpha}{L_c^{\,2}}t$$

Thermal diffusivity:

$$\alpha \equiv \left(\frac{k}{\rho c}\right) \quad (m^2\ s^{-1})$$

Define Fo as the Fourier number (dimensionless time)

$$Fo \equiv \frac{\alpha}{L_c^{\,2}} t \quad \text{and Biot number} \quad Bi \equiv \frac{hL_C}{k}$$

The temperature variation can be expressed as

$$\mathbf{T} = \exp(-Bi*Fo)$$

where L_c is a characteristic length scale : realte to the size of the solid invloved in the problem

for example , $L_c = \frac{r_o}{2}$ (half - radius) when the solid is a cylinder.

$L_c = \frac{r_o}{3}$ (one - third radius) when the solid is sphere

$L_c = L$ (half thickness) when the solid is aplane wall with a 2L thickness

4- Analytical Solution.

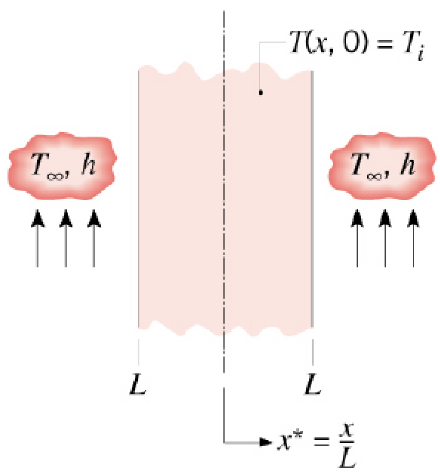
The Plane Wall: Solution to the Heat Equation for a Plane Wall with Symmetrical Convection Conditions

$$\frac{1}{a} \cdot \frac{\partial T}{\partial \tau} = \frac{\partial^2 T}{\partial x^2}$$

$$T(x,0) = T_i$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$$

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=l} = h\left[T(L,t) - T_\infty\right]$$



Note: Once spatial variability of temperature is included, there is existence of seven different independent variables.

How may the functional dependence be simplified?

- The answer is **Non-dimensionalisation.** We first need to understand the physics behind the phenomenon, identify parameters governing the process, and group them into meaningful non-dimensional numbers.

Dimensionless temperature difference: $\theta^* = \dfrac{\theta}{\theta_i} = \dfrac{T - T_\infty}{T_i - T_\infty}$

Dimensionless coordinate: $x^* = \dfrac{x}{L}$

Dimensionless time: $t^* = \dfrac{\alpha t}{L^2} = Fo$

The Biot Number: $Bi = \dfrac{hL}{k_{solid}}$

The solution for temperature will now be a function of the other non-dimensional quantities

$$\theta^* = f(x^*, Fo, Bi)$$

Exact Solution: $\theta^* = \sum_{n=1}^{\infty} C_n \exp\left(-\zeta_n^2 Fo\right)\cos\left(\zeta_n x^*\right)$

$$C_n = \frac{4\sin\zeta_n}{2\zeta_n + \sin(2\zeta_n)} \qquad \zeta_n \tan\zeta_n = Bi$$

The roots (eigenvalues) of the equation can be obtained from tables given in standard textbooks.

The One-Term Approximation $Fo > 0.2$

Variation of mid-plane temperature with time Fo $(x^* = 0)$

$$\theta_0^* = \frac{T - T_\infty}{T_i - T_\infty} \approx C_1 \exp\left(-\zeta_1^2 Fo\right)$$

From tables given in standard textbooks, one can obtain C_1 and ζ_1 as a function of Bi.

Variation of temperature with location (x^*) and time (Fo):

$$\theta^* = \theta_0^* = \cos\left(\zeta_1 x^*\right)$$

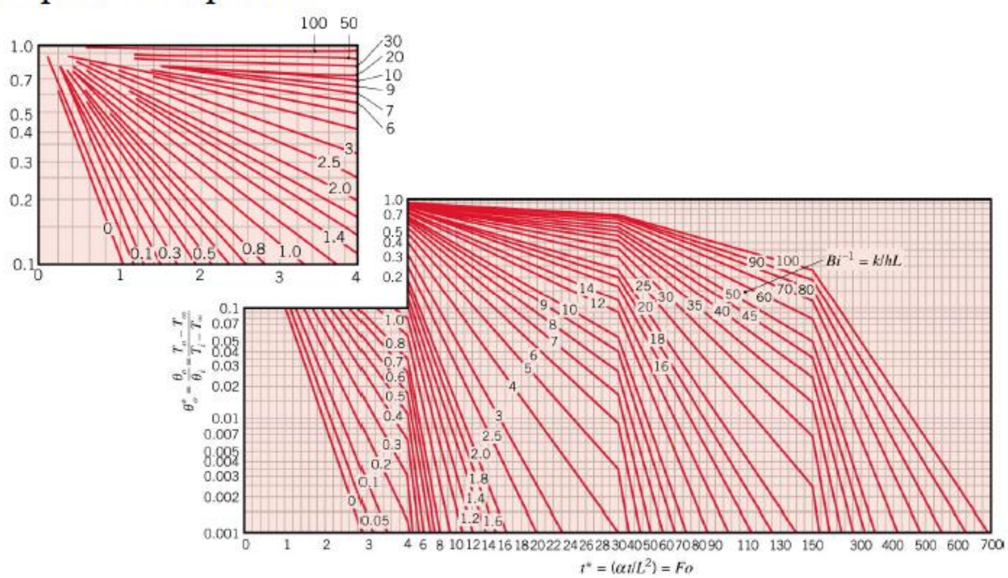
Change in thermal energy storage with time: $\Delta E_{st} = -Q$

$$Q = Q_0\left(1 - \frac{\sin \zeta_1}{\zeta_1}\right)\theta_0^*$$

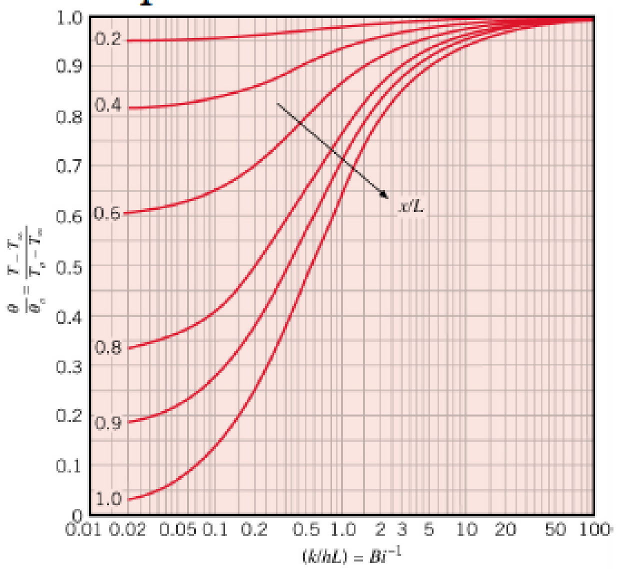
$$Q_0 = \rho c V\left(T_i - T_\infty\right)$$

5- Graphical Representation (The Heisler Charts).

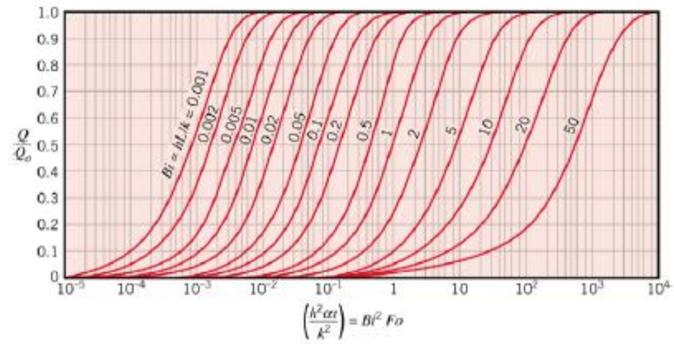
Midplane Temperature:



Temperature Distribution



Change in Thermal Energy Storage



Assumptions in using Heisler charts:

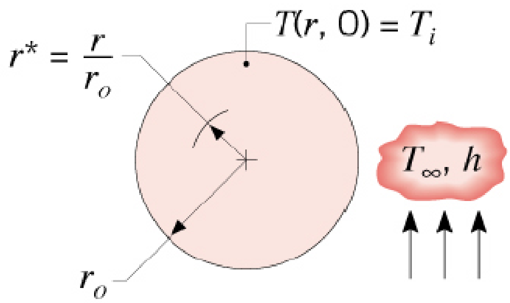
- Constant Ti and thermal properties over the body
- Constant boundary fluid T∞ by step change
- Simple geometry: slab, cylinder or sphere

Lecture Nine

Unsteady Radial System

1- Radial System Coordinate.

Long Rods or Spheres Heated or Cooled by convection mechanism.

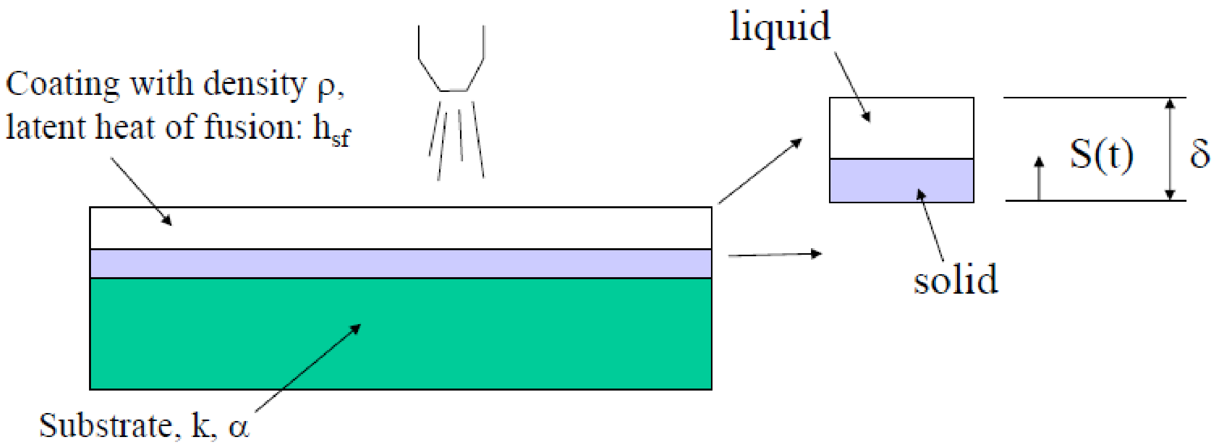


Similar Heisler charts are available for radial systems in standard text books.

Important tips: Pay attention to the length scale used in those charts, and calculate your Biot number accordingly.

2- Unsteady Heat Transfer in Semi-infinite Solids.

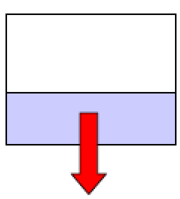
❑ Solidification process of the coating layer during a thermal spray operation is an unsteady heat transfer problem. As we discuss earlier, thermal spray process deposits thin layer of coating materials on surface for protection and thermal resistant purposes, as shown. The heated, molten materials will attach to the substrate and cool down rapidly. The cooling process is important to prevent the accumulation of residual thermal stresses in the coating layer.



Example

- As described in the previous slide, the cooling process can now be modeled as heat loss through a semi-infinite solid. (Since the substrate is significantly thicker than the coating layer) The molten material is at the fusion temperature T_f and the substrate is maintained at a constant temperature T_i. Derive an expression for the total time that is required to solidify the coating layer of thickness δ.

- Assume the molten layer stays at a constant temperature T_f throughout the process. The heat loss to the substrate is solely supplied by the release of the latent heat of fusion.



Heat transfer from the molten material to the substrate (q=q"A)

From energy balance:

$h_{sf}\Delta m$ (solidified mass during Δt) $= \Delta Q = q'' A \Delta t$ (energy input)

$$h_{sf}\frac{dm}{dt} = q''A, \text{ where } m = \rho V = \rho AS,$$

where S is solidified thickness

$$\rho\frac{dS}{dt} = q''$$

❑ Identify that the previous situation corresponds to the case of a semi-infinite transient heat transfer problem with a constant surface temperature boundary condition. This boundary condition can be modeled as a special case of convection boundary condition case by setting h=∞, therefore, $T_s=T_\infty$).

If the surface temperature is T_s and the initial temperature of the bolck is T_i, the analytical solution of the problem can be found: The temperature distribution and the heat transfer into the block are:

$\frac{T(x,t)-T_s}{T_i-T_s}=erf\left(\frac{x}{2\sqrt{\alpha t}}\right)$, where erf( ) is the Gaussian error function.

It is defined as $\text{erf}(w)=\frac{2}{\sqrt{\pi}}\int_0^w e^{-v^2}dv$

$$q_s''(t)=\frac{k(T_s-T_i)}{\sqrt{\pi\alpha t}}$$

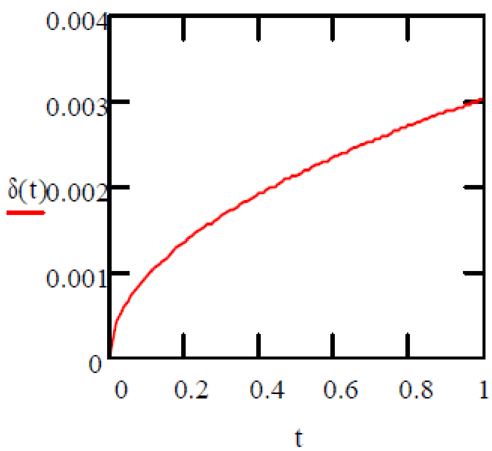
From the previous equation

$\rho h_{sf}\frac{dS}{dt}=q''=\frac{k(T_f-T_i)}{\sqrt{\pi\alpha t}}$, and $\int_0^\delta dS=\frac{k(T_f-T_i)}{\rho h_{sf}\sqrt{\pi\alpha}}\int_0^t\frac{dt}{\sqrt{t}}$

$\delta(t)=\frac{2k(T_f-T_i)}{\rho h_{sf}\sqrt{\pi\alpha}}\sqrt{t}$, therefore, $\delta\propto\sqrt{t}$. Cooling time $t=\frac{\pi\alpha}{4k^2}\left(\frac{\delta\rho h_{sf}}{T_f-T_i}\right)^2$

❑ Use the following values to calculate: k=120 W/m.K, $\alpha=4\times10^{-5}$ m²/s, ρ=3970 kg/m³, and $h_{sf}=3.577\times10^6$ J/kg, T_f=2318 K, T_i=300K, and δ=2 mm

$$\delta(t)=\frac{2k(T_f-T_i)}{\rho h_{sf}\sqrt{\pi\alpha}}\sqrt{t}=0.00304\sqrt{t}$$



- $\delta(t) \propto t^{1/2}$
- Therefore, the layer solidifies very fast initially and then slows down as shown in the figure
- Note: we neglect contact resistance between the coating and the substrate and assume temperature of the coating material stays the same even after it solidifies.

- To solidify 2 mm thickness, it takes 0.43 seconds.

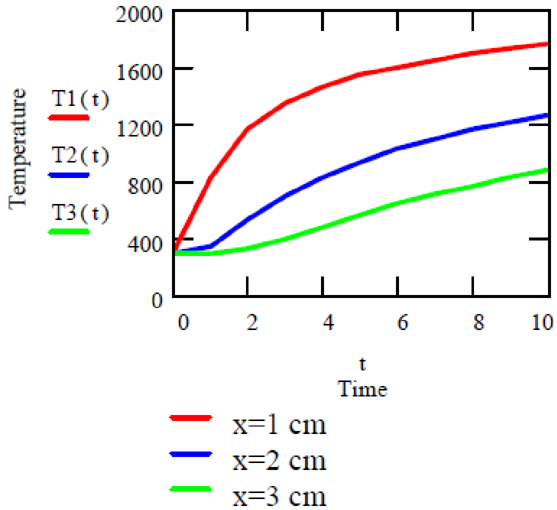
- What will be the substrate temperature as it varies in time? The temperature distribution is:

$$\frac{T(x,t)-T_S}{T_i-T_S}=erf\left(\frac{x}{2\sqrt{\alpha t}}\right),$$

$$T(x,t)=2318+(300-2318)erf\left(\frac{x}{2\sqrt{\alpha t}}\right)=2318-2018erf\left(79.06\frac{x}{\sqrt{t}}\right)$$

- For a fixed distance away from the surface, we can examine the variation of the temperature as a function of time. Example, 1 cm deep into the substrate the temperature should behave as:

$$T(x=0.01,t)=2318-2018erf\ 79.06\frac{x}{\sqrt{t}}\ =2318-2018erf\ \frac{0.79}{\sqrt{t}}$$

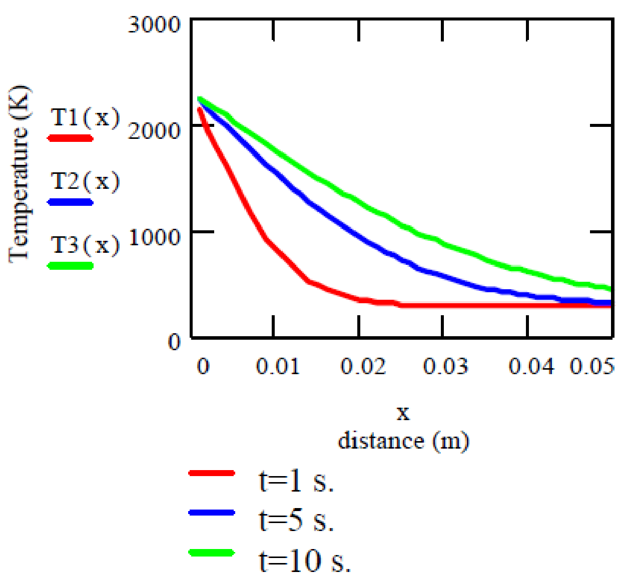


❑ At x=1 cm, the temperature rises almost instantaneously at a very fast rate. A short time later, the rate of temp. increase slows down significantly since the energy has to distribute to a very large mass.

❑ At deeper depth (x=2 & 3 cm), the temperature will not respond to the surface condition until much later.

❑ We can also examine the spatial temperature distribution at any given time, say at t=1 second.

$$T(x,t=1)=2318-2018\,erf\;79.06\frac{x}{\sqrt{t}}\;=2318-2018\,erf\;79.06x$$



❑ Heat penetrates into the substrate as shown for different time instants.

❑ It takes more than 5 seconds for the energy to transfer to a depth of 5 cm into the substrate

❑ The slopes of the temperature profiles indicate the amount of conduction heat transfer at that instant.

Lecture Ten

Numerical Methods for Unsteady Heat Transfer

❑ Unsteady heat transfer equation, no generation, constant k, two-dimensional in Cartesian coordinate:

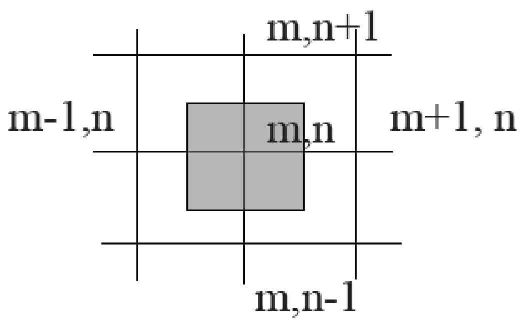
$$\frac{1}{\alpha}\frac{\partial T}{\partial t}=\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}$$

❑ We have learned how to discretize the Laplacian operator into system of finite difference equations using nodal network. For the unsteady problem, the temperature variation with time needs to be discretized too. To be consistent with the notation from the book, we choose to analyze the time variation in small time increment Δt, such that the real time t=pΔt. The time differentiation can be approximated as:

$$\left.\frac{\partial T}{\partial t}\right|_{m,n}\approx\frac{T_{m,n}^{P+1}-T_{m,n}^{P}}{\Delta t},$$ while m & n correspond to nodal location

such that x=mΔx, and y=nΔy as introduced earlier.

1- Finite Difference Equation.



❑ From the nodal network to the left, the heat equation can be written in finite difference form:

$$\frac{1}{\alpha}\frac{T_{m,n}^{P+1}-T_{m,n}^{P}}{\Delta t}=\frac{T_{m+1,n}^{P}+T_{m-1,n}^{P}-2T_{m,n}^{P}}{(\Delta x)^2}+\frac{T_{m,n+1}^{P}+T_{m,n-1}^{P}-2T_{m,n}^{P}}{(\Delta y)^2}$$

Assume Δx=Δy and the discretized Fourier number Fo= $\frac{\alpha\Delta t}{(\Delta x)^2}$

$$T_{m,n}^{P+1}=Fo\left(T_{m+1,n}^{P}+T_{m-1,n}^{P}+T_{m,n+1}^{P}+T_{m,n-1}^{P}\right)+(1-4Fo)T_{m,n}^{P}$$

This is the **explicit**, finite difference equation for a 2-D, unsteady heat transfer equation.

The temperature at time p+1 is explicitly expressed as a function of neighboring temperatures at an earlier time p

❑ Some common nodal configurations are listed in table for your reference. On the third column of the table, there is a stability criterion for each nodal configuration. This criterion has to be satisfied for the finite difference solution to be stable. Otherwise, the solution may be diverging and never reach the final solution.

❑ For example, Fo≤1/4. That is, $\alpha\Delta t/(\Delta x)^2 \le 1/4$ and $\Delta t \le (1/4\alpha)(\Delta x)^2$. Therefore, the time increment has to be small enough in order to maintain stability of the solution.

❑ This criterion can also be interpreted as that we should require the coefficient for $T^{P}_{m,n}$ in the finite difference equation be greater than or equal to zero.

❑ Question: Why this can be a problem? Can we just make time increment as small as possible to avoid it?

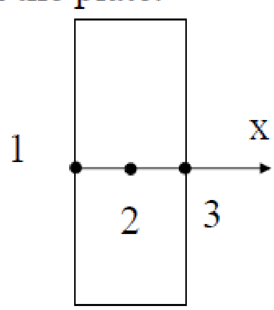
2- **Finite Difference Solution.**

Question: How do we solve the finite difference equation derived?

- First, by specifying initial conditions for all points inside the nodal network. That is to specify values for all temperature at time level p=0.
- Important: check stability criterion for each points.
- From the explicit equation, we can determine all temperature at the next time level p+1=0+1=1. The following transient response can then be determined by marching out in time p+2, p+3, and so on.

Example

- Example: A flat plate at an initial temperature of 100 deg. is suddenly immersed into a cold temperature bath of 0 deg. Use the unsteady finite difference equation to determine the transient response of the temperature of the plate.



L(thickness)=0.02 m, k=10 W/m.K, $\alpha=10\times10^{-6}$ m²/s, h=1000 W/m².K, T_i=100°C, T_∞=0°C, Δx=0.01 m

$Bi=(h\Delta x)/k=1$, $Fo=(\alpha\Delta t)/(\Delta x)^2=0.1$

There are three nodal points: 1 interior and two exterior points: For node 2, it satisfies the case 1 configuration in table.

$$T_2^{P+1} = Fo(T_1^P + T_3^P + T_2^P + T_2^P) + (1-4Fo)T_2^P = Fo(T_1^P + T_3^P) + (1-2Fo)T_2^P$$
$$= 0.1(T_1^P + T_3^P) + 0.8T_2^P$$

Stability criterion: $1-2Fo \ge 0$ or $Fo=0.1 \le \frac{1}{2}$, it is satisfied

For nodes 1 & 3, they are consistent with the case 3 in table.

Node 1: $T_1^{P+1} = Fo(2T_2^P + T_1^P + T_1^P + 2BiT_\infty) + (1 - 4Fo - 2BiFo)T_1^P$

$= Fo(2T_2^P + 2BiT_\infty) + (1 - 2Fo - 2BiFo)T_1^P = 0.2T_2^P + 0.6T_1^P$

Node 3: $T_3^{P+1} = 0.2T_2^P + 0.6T_3^P$

Stability criterion: (1-2Fo-2BiFo) ≥ 0, $\frac{1}{2} \geq Fo(1 + Bi) = 0.2$ and it is satisfied

System of equations

$T_1^{P+1} = 0.2T_2^P + 0.6T_1^P$

$T_2^{P+1} = 0.1(T_1^P + T_3^P) + 0.8T_2^P$

$T_3^{P+1} = 0.2T_2^P + 0.6T_3^P$

Use initial condition, $T_1^0 = T_2^0 = T_3^0 = 100$,

$T_1^1 = 0.2T_2^0 + 0.6T_1^0 = 80$

$T_2^1 = 0.1(T_1^0 + T_3^0) + 0.8T_2^0 = 100$

$T_3^1 = 0.2T_2^0 + 0.6T_3^0 = 80$

Marching in time, $T_1^1 = T_3^1 = 80$, $T_2^1 = 100$

$T_1^2 = 0.2T_2^1 + 0.6T_1^1 = 0.2(100) + 0.6(80) = 68$

$T_2^2 = 0.1(T_1^1 + T_3^1) + 0.8T_2^1 = 0.1(80 + 80) + 0.8(100) = 96$

$T_3^2 = 0.2T_2^1 + 0.6T_3^1 = 0.2(100) + 0.6(80) = 68$, and so on